§7.02

#3. $\int t^2\cos t dt$. $\begin{array}{rcl} u &=& t^2 \\ du &=& 2t\,dt \end{array}$ $\quad \begin{array}{rcl} dw &=& \cos t\,dt \\ w &=& \sin t \end{array}$ Hence $\int t^2\cos t\,dt = t^2\sin t - \int(\sin t)2t\,dt = t^2\sin t - 2\int t\sin t dt$. Apply "Parts" to $\int t\sin t dt$ with $\begin{array}{rcl} u &=& t \\ du &=& dt \end{array}$ $\quad \begin{array}{rcl} dw &=& \sin t\,dt \\ w &=& -\cos t \end{array}$ so $\int t\sin t\,dt = t(-\cos t) - \int(-\cos t)dt = -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + C$. Putting these two results together, we get $\int t^2\cos t dt = t^2(\sin t) - 2\Big(-t\cos t + \sin t\Big) + C = t^2(\sin t) + 2t\cos t - 2\sin t + C$.

#5. $\int_1^2 x\ln x\,dx$. Try $u = x$, $dw = \ln x\,dx$. To do this, we need $\int dw = \int \ln x\,dx$ and this can be done by parts or gotten from the book: $\int \ln x\,dx = x\ln x - x + C$, so $w = x\ln x - x$ and $du = dx$. Then $\int_1^2 x\ln x\,dx = x(x\ln x - x)\Big|_1^2 - \int_1^2 (x\ln x - x)dx = \big(2(2\ln 2 - 2)\big) - \big(1(1\ln 1 - 1)\big) + \int_1^2 x\,dx - \int_1^2 x\ln x\,dx = 4\ln 2 - 4 - 1(1\cdot 0 - 1) + \frac{x^2}{2}\Big|_1^2 - \int_1^2 x\ln x\,dx = 4\ln 2 - 4 + 1 + \frac{2^2}{2} - \frac{1}{2} - \int_1^2 x\ln x\,dx = 4\ln 2 - 3 + 2 - \frac{1}{2} - \int_1^2 x\ln x\,dx = 4\ln 2 - 1 - \frac{1}{2} - \int_1^2 x\ln x\,dx = 4\ln 2 - \frac{3}{2} - \int_1^2 x\ln x\,dx$. Solve for $\int_1^2 x\ln x\,dx$: $2\int_1^2 x\ln x\,dx = 4\ln 2 - \frac{3}{2}$ so $\int_1^2 x\ln x\,dx = 2\ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$.

#7. $\int \arctan y dy$. $\begin{array}{rcl} u &=& \arctan y \\ du &=& \frac{dy}{1+y^2} \end{array}$ $\quad \begin{array}{rcl} dw &=& dy \\ w &=& y \end{array}$ so $\int \arctan y dy = y\arctan y - \int \frac{y}{1+y^2}dy$. This last integral can be done by substitution $u = y^2 + 1$, $du = 2y\,dy$, so $\int \frac{y}{1+y^2}\,dy = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(1+y^2) + C$ so $\int \arctan y dy = y\arctan y - \frac{1}{2}\ln(1+y^2) + C$.

#13. $\int (x^2 - 5x)e^x dx = \int x^2e^x dx - 5\int xe^x dx$. Do $\int x^2e^x dx$ by parts $\begin{array}{rcl} u &=& x^2 \\ du &=& 2x\,dx \end{array}$ $\quad \begin{array}{rcl} dw &=& e^x dx \\ w &=& e^x \end{array}$: so $\int x^2e^x dx = x^2e^x - 2\int xe^x dx$. Do $\int xe^x dx$ by parts: $\begin{array}{rcl} u &=& x \\ du &=& dx \end{array}$ $\quad \begin{array}{rcl} dw &=& e^x dx \\ w &=& e^x \end{array}$ so $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$. Hence $\int x^2e^x dx = x^2e^x - 2\big(xe^x - e^x\big) + C = x^2e^x - 2xe^x + 2e^x + C$. Finally $\int (x^2 - 5x)e^x dx = (x^2e^x - 2xe^x + 2e^x) - 5(xe^x - e^x) + C = x^2e^x - 7xe^x + 7e^x + C = (x^2 - 7x + 7)e^x + C$.

#21. $\int e^\theta \sin\theta d\theta$. $\begin{array}{rcl} u &=& e^\theta \\ du &=& e^\theta d\theta \end{array}$ $\quad \begin{array}{rcl} dw &=& \sin\theta\,d\theta \\ w &=& -\cos\theta \end{array}$ so $\int e^\theta\sin\theta d\theta = -e^\theta\cos\theta - \Big(\int(-\cos\theta)e^\theta d\theta\Big) = -e^\theta\cos\theta + \int e^\theta\cos\theta d\theta$. We try $\int e^\theta\cos\theta d\theta$ by parts also: $\begin{array}{rcl} u &=& e^\theta \\ du &=& e^\theta d\theta \end{array}$ $\quad \begin{array}{rcl} dw &=& \cos\theta\,d\theta \\ w &=& \sin\theta \end{array}$

so $\int e^\theta \cos\theta \, d\theta = e^\theta \sin\theta - \int e^\theta \sin\theta \, d\theta$. Hence $\int e^\theta \sin\theta \, d\theta = -e^\theta \cos\theta + \big(e^\theta \sin\theta - \int e^\theta \sin\theta \, d\theta\big)$. Solving for $\int e^\theta \sin\theta \, d\theta$ yields $2\int e^\theta \sin\theta \, d\theta = -e^\theta \cos\theta + e^\theta \sin\theta$, so $\int e^\theta \sin\theta \, d\theta = \frac{(\sin\theta - \cos\theta)}{2} e^\theta + C$.

#27. $\int_0^{\pi/3} x \tan^2 x \, dx$. Parts would be nice but with $u = x$ we will need $\int \tan^2 x \, dx$. We know $\int \sec^2 x \, dx$ and $\tan^2 = \sec^2 - 1$ so $\int \tan^2 x \, dx = \int \sec^2 x \, dx - \int dx = \tan x - x + C$. Now by parts, $\begin{array}{rcl} u &=& x \\ du &=& dx \end{array} \quad \begin{array}{rcl} dw &=& \tan^2 x \, dx \\ w &=& \tan x - x \end{array}$ so $\int_0^{\pi/3} x \tan^2 x \, dx = x(\tan x - x)\Big|_0^{\pi/3} - \int (\tan x - x) \, dx = \Big(\frac{\pi}{3}\big(\tan(\pi/3) - \pi/3\big)\Big) - \Big(0\big(\tan(0) - 0\big)\Big) - \Big(\ln|\sec x|\Big|_0^{\pi/3} - \frac{x^2}{2}\Big|_0^{\pi/3}\Big) = \frac{\pi}{3}(\sqrt{3} - \frac{\pi}{3}) - \Big(\ln|\sec(\pi/3)| - \frac{(\pi/3)^2}{2} - \ln|\sec(0)| - 0\Big) = \frac{\pi\sqrt{3}}{3} - \frac{\pi^2}{9} - (\ln 2 - \ln 1) = \frac{\pi\sqrt{3}}{3} - \frac{\pi^2}{9} + \frac{\pi^2}{18} - \ln 2 = \frac{\pi\sqrt{3}}{3} - \frac{\pi^2}{18} - \ln 2$. Another way to proceed is to make the substitution $z = \tan x$, $dz = \sec^2 x dx = (1+z^2) dx$ so $dx = \frac{dz}{1+z^2}$. Hence $\int_0^{\pi/3} x \tan^2 x \, dx = \int_0^{\sqrt{3}} (\arctan z) \frac{z^2}{1+z^2} \, dz$. Now $\begin{array}{rcl} u &=& \arctan z \\ du &=& \frac{dz}{1+z^2} \end{array} \quad \begin{array}{rcl} dw &=& \frac{z^2 dz}{1+z^2} \\ w &=& ? \end{array}$. To find w integrate $\int \frac{z^2}{1+z^2} dz = \int 1 dz - \int \frac{dz}{1+z^2} = z - \arctan z + C$ so take $w = z - \arctan z$. Then $\int_0^{\sqrt{3}} (\arctan z) \frac{z^2}{1+z^2} \, dz = (\arctan z)(z - \arctan z)\Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} (z - \arctan z) \frac{dz}{1+z^2} = \frac{\pi}{3}(\sqrt{3} - \frac{\pi}{3}) - 0 - \int_0^{\sqrt{3}} \frac{z dz}{1+z^2} + \int_0^{\sqrt{3}} \frac{\arctan z \, dz}{1+z^2}$. Now $\int_0^{\sqrt{3}} \frac{z dz}{1+z^2} = \frac{1}{2} \ln(1+z^2)\Big|_0^{\sqrt{3}} = \frac{1}{2} \ln 4 = \ln 2$ and $\int_0^{\sqrt{3}} \frac{\arctan z \, dz}{1+z^2}$ can be integrated by the substitution $w = \arctan z$, $dw = \frac{dz}{1+z^2}$ so $\int_0^{\sqrt{3}} \frac{\arctan z \, dz}{1+z^2} = \int_0^{\pi/3} w \, dw = \frac{w^2}{2}\Big|_0^{\pi/3} = \frac{\pi^2}{18}$. The pieces fit together as above.

#29. $\int \sin(\ln x) dx$. One way is by parts: $\begin{array}{rcl} u &=& \sin(\ln x) \\ du &=& \frac{\cos(\ln x) \, dx}{x} \end{array} \quad \begin{array}{rcl} dw &=& dx \\ w &=& x \end{array}$ so $\int \sin(\ln x) dx = x \sin(\ln x) - \int x \frac{\cos(\ln x) \, dx}{x} = x \sin(\ln x) - \int \cos(\ln x) \, dx$. Repeat the procedure for $\int \cos(\ln x) \, dx$: $\begin{array}{rcl} u &=& \cos(\ln x) \\ du &=& \frac{-\sin(\ln x) \, dx}{x} \end{array} \quad \begin{array}{rcl} dw &=& dx \\ w &=& x \end{array}$ so $\int \cos(\ln x) \, dx = x \cos(\ln x) - \int x \frac{-\sin(\ln x) \, dx}{x} = x \cos(\ln x) + \int \sin(\ln x) \, dx$. Hence $\int \sin(\ln x) dx = x \sin(\ln x) - \big(x \cos(\ln x) + \int \sin(\ln x) \, dx\big) = x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) \, dx$ and we can solve for $\int \sin(\ln x)dx$: $2\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x)$ so $\int \sin(\ln x)dx = \frac{x(\sin(\ln x)-\cos(\ln x))}{2} + C$. A second approach. Substitute $u = \ln x$, $du = \frac{dx}{x}$, so $dx = xdu = e^u du$. Hence $\int \sin(\ln x)dx = \int e^u \sin u du$ and we have seen how to integrate this integral using parts twice.

#47. The two formulae are both correct since $\sin(\arccos x) = \sqrt{1-x^2}$.