§7.03

#1. $\dfrac{5x-13}{(x-3)(x-2)} = \dfrac{A}{x-3} + \dfrac{B}{x-2}$: $5x-13 = A(x-2)+B(x-3)$: plug in $x=2$, $-3 = B(-1)$ or $B=3$; plug in $x=3$, $2 = A(1)$ or $\dfrac{5x-3}{(x-3)(x-2)} = \dfrac{2}{x-3} + \dfrac{3}{x-2}$.

#5. $\dfrac{z+1}{z^2(z-1)} = \dfrac{A}{z} + \dfrac{B}{z^2} + \dfrac{C}{z-1}$: $z+1 = Az(z-1)+B(z-1)+Cz^2$. Plug in $z=0$, $1 = B(-1)$ so $B=-1$. Take derivative and plug in $z=0$; $1 = A(-1)+B$ so $A=-2$. Plug in $z=1$; $2 = C(1)^2$ so $C=2$ and $\dfrac{z+1}{z^2(z-1)} = \dfrac{-2}{z} + \dfrac{-1}{z^2} + \dfrac{2}{z-1}$.

#7. $\dfrac{t^2+8}{t^2-5t+6}$. First note that degree of numerator equals degree of denominator so polynomial long division needs to be used: $\dfrac{t^2+8}{t^2-5t+6} = 1 + \dfrac{5t+2}{t^2-5t+6}$. The denominator is reducible, $t^2-5t+6 = (t-2)(t-3)$; $\dfrac{5t+2}{t^2-5t+6} = \dfrac{A}{t-2} + \dfrac{B}{t-3}$ or $5t+2 = A(t-3)+B(t-2)$. Plug in $t=2$; $12 = A(-1)$ so $A=-12$. Plug in $t=3$; $17 = B(1)$ so $B=17$. Hence $\dfrac{t^2+8}{t^2-5t+6} = 1 + \dfrac{-12}{t-2} + \dfrac{17}{t-3}$.

#11. $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx$. Apply partial fractions to the integrand. The quadratic is reducible $x^2+5x-6 = (x+6)(x-1)$ so $\dfrac{x+4}{x^2+5x-6} = \dfrac{A}{x+6} + \dfrac{B}{x-1}$ or $x+4 = A(x-1)+B(x+6)$. Plug in $x=1$; $5 = B(7)$ so $B = \frac{5}{7}$. Plug in $x=-6$; $-2 = A(-7)$ so $A = \frac{2}{7}$ and $\dfrac{x+4}{x^2+5x-6} = \frac{1}{7}\left(\dfrac{2}{x+6} + \dfrac{5}{x-1}\right)$. Hence $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C$.

#17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2+2x+1}$. Apply partial fractions to the integrand. First do polynomial long division: $\dfrac{x^3}{x^2+2x+1} = x-2+\dfrac{3x+2}{x^2+2x+1}$. The quadratic is reducible $x^2+2x+1 = (x+1)^2$ so $\dfrac{3x+2}{x^2+2x+1} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2}$ or $3x+2 = A(x+1)+B$. Plug in $x=-1$, $-1 = B$. Take derivatives, $3 = A$. Hence $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2+2x+1} = \int_0^1 (x-2)\,dx + \int_0^1 \frac{3dx}{x+1} + \int_0^1 \frac{-dx}{(x+1)^2} = \frac{x^2}{2}\Big|_0^1 - 2x\Big|_0^1 + 3\ln|x+1|\Big|_0^1 - \frac{(x+1)^{-1}}{-1}\Big|_0^1 = \frac{1}{2} - 0 - (2-0) + 3(\ln 2 - \ln 1) - \left(-\frac{1}{2} - (-1)\right) = \frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2} - 1 = 3\ln 2 - 2$.

#21. $\displaystyle\int_0^1 \frac{dx}{(x+1)(x^2+1)}$. Partial fractions, $\dfrac{1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$ or $1 = A(x^2+1)+(Bx+C)(x+1)$. Plug in $x=-1$; $1 = A\big((-1)^1+1\big)$ or $A = \frac{1}{2}$. Plug in $x=0$; $1 = A(1)+C(1)$ so $C = \frac{1}{2}$. Plug in $x=1$; $1 = A(2)+(B+C)(2)$ so $1 = 1+2B+1$ or $B = -\frac{1}{2}$. Hence $\displaystyle\int_0^1 \frac{dx}{(x+1)(x^2+1)} = \frac{1}{2}\int_0^1 \frac{dx}{x+1} - \frac{1}{2}\int_0^1 \frac{x-1}{x^2+1}dx$

Work on $\int_0^1 \frac{x-1}{x^2+1}dx = \int_0^1 \frac{xdx}{x^2+1} + \int_0^1 \frac{dx}{x^2+1}$. Work on $\int_0^1 \frac{xdx}{x^2+1}$ by substituting $u = x^2+1$ $du = 2xdx$ so $\int_0^1 \frac{xdx}{x^2+1} = \frac{1}{2}\int_1^2 \frac{du}{u} = \frac{1}{2}\ln|u|\Big|_1^2 = \frac{1}{2}\ln 2$. $\int_0^1 \frac{dx}{x^2+1} = \arctan x\Big|_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4} - 0$, so $\int_0^1 \frac{x-1}{x^2+1}dx = \frac{1}{2}\ln 2 + \frac{\pi}{4}$. Finally $\int_0^1 \frac{dx}{(x+1)(x^2+1)} = \frac{1}{2}\int_0^1 \frac{dx}{x+1} - \frac{1}{2}(\frac{\ln 2}{2} + \frac{\pi}{4}) = \frac{1}{2}\ln|x+1|\Big|_0^1 - \frac{\ln 2}{4} + \frac{\pi}{8} = \frac{1}{2}\ln 2 - \frac{\ln 2}{4} - \frac{\pi}{8} = \frac{\ln 2}{4} + \frac{\pi}{8}$.

#25. $\int \frac{2s+2}{(s^2+1)(s-1)^3}ds$. Partial fractions yields $\frac{2s+2}{(s^2+1)(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{Ds+E}{s^2+1}$ or $2s+2 = A(s-1)^2(s^2+1) + B(s-1)(s^2+1) + C(s^2+1) + (Ds+E)(s-1)^3$. Plug in $s=1$; $4 = C(2)$ so $C = 2$. Hence $2s+2-2s^2-2 = A(s-1)^2(s^2+1) + B(s-1)(s^2+1) + (Ds+E)(s-1)^3$ and we can divide both sides by $s-1$ to get $-2s = A(s-1)(s^2+1) + B(s^2+1) + (Ds+E)(s-1)^2$. Plug in $s=1$ again; $-2 = B(2)$ so $B = -1$. Again $-2s + (s^2+1) = A(s-1)(s^2+1) + (Ds+E)(s-1)^2$. Divide by $s-1$ again to get $s-1 = A(s^2+1) + (Ds+E)(s-1)$. Plug in $s=1$ again; $0 = A(2)$ so $A = 0$. Divide by $s-1$ again to get $1 = Ds+E$ so $D = 0$ and $E = 1$. Thus $\frac{2s+2}{(s^2+1)(s-1)^3} = \frac{-1}{(s-1)^2} + \frac{2}{(s-1)^3} + \frac{1}{s^2+1}$. Then $\int \frac{2s+2}{(s^2+1)(s-1)^3} = -\int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} + \int \frac{ds}{s^2+1} = -\frac{(s-1)^{-1}}{-1} + 2\Big(\frac{(s-1)^{-2}}{-2}\Big) + \arctan s + C = \frac{1}{s-1} - \frac{1}{(s-1)^2} + \arctan s + C$.

#29. $\int \frac{2x^3-2x^2+1}{x^2-x}\,dx$. To apply partial fractions, we first need to do polynomial long division: $\frac{2x^3-2x^2+1}{x^2-x} = 2x + \frac{1}{x^2-x}$. $\frac{1}{x^2-x} = \frac{A}{x} + \frac{B}{x-1}$ or $1 = A(x-1) + Bx$. Plug in $x = 1$; $1 = B$. Plug in $x = 0$; $1 = A(-1)$ so $A = -1$. Hence $\int \frac{2x^3-2x^2+1}{x^2-x}\,dx = \int 2x\,dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C$.

#33. $\int \frac{y^4+y^2-1}{y^3+y}dy$ Applying polynomial long division $\frac{y^4+y^2-1}{y^3+y} = y - \frac{1}{y^3+y}$ and $\frac{1}{y^3+y} = \frac{A}{y} + \frac{(By+C)}{y^2+1}$ or $1 = A(y^2+1) + (By+C)y$. Plug in $y = 0$; $1 = A(1)$ so $A = 1$. Then $1 - (y^2+1) = (By+C)y$ or $-y^2 = (By+C)y$. Hence $By + C = -y$ so $B = -1$ and $C = 0$. $\int \frac{y^4+y^2-1}{y^3+y}dy = \int y\,dy - \Big(\int \frac{dy}{y} - \int \frac{y\,dy}{y^2+1}\Big) = \frac{y^2}{2} - \big(\ln|y| - \frac{1}{2}\ln|y^2+1|\big) + C = \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln|y^2+1| + C$.

#35. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2}$. Substitute $u = e^t$, $du = e^t dt$ so $\int \frac{e^t\,dt}{e^{2t}+3e^t+2} = \int \frac{du}{u^2+3u+2} =$

$\int \frac{du}{\left(u+\frac{3}{2}\right)^2 - \frac{1}{4}}$. Substitute again $\frac{1}{2}w = \left(u + \frac{3}{2}\right)$, $\frac{1}{2}dw = du$ or $\frac{1}{2}dw = du$. Hence $\int \frac{du}{\left(u+\frac{3}{2}\right)^2 - \frac{1}{4}} = \int \frac{\frac{1}{2}dw}{\frac{1}{4}w^2 - \frac{1}{4}} = \int \frac{2dw}{w^2-1} = \int \frac{dw}{w-1} - \int \frac{dw}{w+1} = \ln|w-1| - \ln|w+1| + C = \left|\frac{w-1}{w+1}\right| + C$. Working back, $\left|\frac{w-1}{w+1}\right| = \left|\frac{2\left(u+\frac{3}{2}\right)-1}{2\left(u+\frac{3}{2}\right)+1}\right| + C = \left|\frac{2u+2}{2u+4}\right| + C = \left|\frac{u+1}{u+2}\right| + C = \left|\frac{e^t+1}{e^t+2}\right| + C.$

#39. $\int \frac{(x-2)^2 \arctan(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2} dx = \int \frac{(x-2)^2 \arctan(2x)}{(4x^2+1)(x-2)^2} dx \int \frac{-12x^3-3x}{(4x^2+1)(x-2)^2} dx = \int \frac{\arctan(2x)}{(4x^2+1)}\, dx - 3\int \frac{4x^3+x}{(4x^2+1)(x-2)^2}\, dx$. First do $\int \frac{\arctan(2x)}{(4x^2+1)}\, dx$ by substitution $w = \arctan(2x)$, $dw = \frac{2dx}{(2x)^2+1}$ so $\int \frac{\arctan(2x)}{(4x^2+1)}\, dx = \frac{1}{2}\int w\, dw = \frac{1}{2}\frac{w^2}{2} + C = \frac{w^2}{4} + C = \frac{\left(\arctan(2x)\right)^2}{4} + C$. Now do $\int \frac{4x^3+x}{(4x^2+1)(x-2)^2}\, dx = \int \frac{x}{(x-2)^2}\, dx$ by partial fractions: $\frac{x}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$ or $x = A(x-2) + B$. Plug in $x = 2$; $2 = B$. Hence $x - 2 = A(x-2)$ or $A = 1$. Hence $\int \frac{x}{(x-2)^2}\, dx = \int \frac{dx}{x-2} + 2\int \frac{dx}{(x-2)^2} = \ln|x-2| + 2\frac{(x-2)^{-1}}{-1} + C = \ln|x-2| - \frac{2}{x-2} + C$. Finally $\int \frac{(x-2)^2 \arctan(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\, dx = \left(\frac{\left(\arctan(2x)\right)^2}{4}\right) - 3\left(\ln|x-2| - \frac{2}{x-2}\right) + C = \frac{\left(\arctan(2x)\right)^2}{4} - 3\ln|x-2| + \frac{6}{x-2} + C.$

#43. $(t^2+2t)\frac{dx}{dt} = 2x + 2$ $(t, x > 0)$, $x(1) = 1$. First work on the differential equation by separation of variables: $\frac{dx}{2x+2} = \frac{dt}{t^2+2t}$. Integrating we get $\int \frac{dx}{2x+2} = \int \frac{dt}{t^2+2t}$. First $\int \frac{dx}{2x+2} = \frac{1}{2}\ln|x+1| + C$. Then by partial fractions $\int \frac{dt}{t^2+2t} = -\frac{1}{2}\int \frac{dt}{t+2} + \frac{1}{2}\int \frac{dt}{t} = -\frac{1}{2}\ln|t+2| + \frac{1}{2}\ln|t| + C$ so we get $\frac{1}{2}\ln|x+1| = -\frac{1}{2}\ln|t+2| + \frac{1}{2}\ln|t| + C$ and our next task is to solve for x as a function of t if we can do it. $\ln|x+1| = \ln|t| - \ln|t+2| + C_0$ so $x + 1 = A_0\frac{t}{t+2}$ or $x = A_0\frac{t}{t+2} - 1$. Finally, since $x(1) = 1$ $1 = A_0\frac{1}{1+2} - 1$ or $1 = A_0\frac{1}{3} - 1$ so $A_0\frac{1}{3} = 2$ or $A_0 = 6$ and $x = \frac{6t}{t+2} - 1$ or $x = \frac{5t-2}{t+2}$.