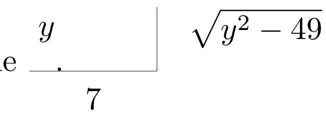
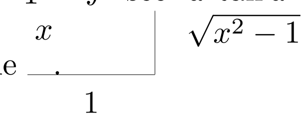
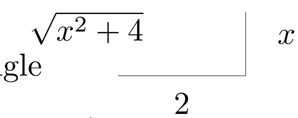
§7.04

\# 3. $\displaystyle\int_{-2}^{2} \frac{dx}{4+x^2} = \frac{1}{2}\arctan(\frac{x}{2})\Big|_{-2}^{2} = \frac{1}{2}\Big(\arctan 1 - \arctan(-1)\Big) = \frac{1}{2}\Big(\frac{\pi}{4} - (-\frac{\pi}{4}\Big) = \frac{\pi}{4}.$

\# 5. $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}} = \arcsin(\frac{x}{3})\Big|_{0}^{3/2} = \arcsin(\frac{1}{2}) - \arcsin 0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$

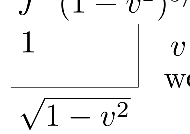
\# 11. $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\,dy$. Substitute $y = 7\sec u$, $dy = 7\sec u\ \tan u\ du$, so $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\,dy = \int \frac{49\tan u \sec u \tan u}{7\sec u}\,du = 7\int \tan^2 u\,du = 7\int (\sec^2 u - 1)du = 7\big(\tan u - u\big) + C = 7\Big(\frac{\sqrt{y^2-49}}{7} - \operatorname{arcsec}(\frac{y}{7})\Big) + C$. Use the triangle 

\# 13. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}$. Substitute $x = \sec u$, $du = \sec u \tan u\,du$, so $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec u\ \tan u}{\sec^2 u\ \tan u}du = \int \frac{1}{\sec u}\,du = \int \cos u\,du = \sin u + C = \frac{\sqrt{x^2-1}}{x} + C$. Use the triangle 

\# 15. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$. Substitute $x = 2\tan u$, $dx = 2\sec^2 u du$, so $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}} = \frac{16\tan^3 u\ \sec^2 u}{2\sec u}du = 8\int \tan^3 u\ \sec u\ du = 8\int \tan^2 u(\tan u\ \sec u)\ du = 8\int (\sec^2 u - 1)(\tan u\ \sec u)\ du.$ Now substitute $w = \sec u$, $dw = \tan u\ \sec u\ dw$, so $\displaystyle 8\int (\sec^2 u - 1)(\tan u\ \sec u)\ du = 8\int (w^2-1)dw = 8\big(\frac{w^3}{3} - w\big) + C$. Furthermore, $w = \sec u$: use the triangle 

so $w = \frac{\sqrt{x^2+4}}{2}$. Hence $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}} = 8\frac{\big(\sqrt{x^2+4}\big)^3}{3\cdot 2^3} - 8\frac{\sqrt{x^2+4}}{2} + C = \frac{\big(x^2+4\big)^{3/2}}{3} - 4\sqrt{x^2+4} + C.$

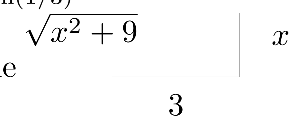
\# 19. $\displaystyle\int_0^{\sqrt{3}/2} \frac{4x^2dx}{(1-x^2)^{3/2}}$. Substitute $x = \sin u$, $dx = \cos u\,du$, so $\displaystyle\int_0^{\sqrt{3}/2} \frac{4x^2dx}{(1-x^2)^{3/2}} = \int_0^{\pi/3} \frac{4\sin^2 u\ \cos u\ du}{\cos^3 u} = 4\int_0^{\pi/3} \tan^2 u\ du = 4\int_0^{\pi/3} \big(\sec^2 u - 1\big)\ du = 4\Big(\tan u - u\Big)\Big|_0^{\pi/3} = 4\big(\tan(\pi/3) - \frac{\pi}{3}\big) - 4\big(\tan 0 - 0\big) = 4\sqrt{3} - \frac{4\pi}{3}.$

\# 27. $\displaystyle\int \frac{v^2dv}{(1-v^2)^{5/2}}$. Substitute $v = \sin u$, $dv = \cos u du$, so $\displaystyle\int \frac{v^2dv}{(1-v^2)^{5/2}} = \int \frac{\sin^2 u\ \cos u}{\cos^5 u}du = \int \tan^2 u\ \sec^2 u\ du.$ Substitute $w = \tan u$, $dw = \sec^2 u\ du$, so $\displaystyle\int \frac{v^2dv}{(1-v^2)^{5/2}} = \int w^2\ dw = \frac{w^3}{3} + C.$ Now $w = \tan u$ and using the triangle 

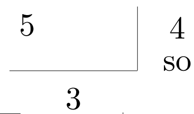
we see

$w = \dfrac{v}{\sqrt{1-v^2}}$, so $\displaystyle\int \frac{v^2 dv}{(1-v^2)^{5/2}} = \tfrac{1}{3}\Big(\frac{v}{\sqrt{1-v^2}}\Big)^3 + C.$

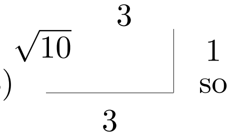
\# 29. $\displaystyle\int_0^{\ln 4} \frac{e^t dt}{\sqrt{e^{2t}+9}}$. Substitute $x = e^t$, $dx = e^t dt$, so $\displaystyle\int_0^{\ln 4} \frac{e^t dt}{\sqrt{e^{2t}+9}} = \int_1^4 \frac{dx}{\sqrt{x^2+9}}$. Now substitute $x = 3\tan u$, $dx = 3\sec^2 u\, du$ so $\displaystyle\int_0^{\ln 4} \frac{e^t dt}{\sqrt{e^{2t}+9}} = \int_{\arctan(1/3)}^{\arctan(4/3)} \frac{3\sec^2 u\, du}{3\sec u} =$ $\displaystyle 3\int_{\arctan(1/3)}^{\arctan(4/3)} \sec u du = \ln|\tan u + \sec u|\Big|_{\arctan(1/3)}^{\arctan(4/3)}$. Use the triangle



Filling in the numbers for arctan(43)



so $\sec\big(\arctan(4/3)\big) = \frac{5}{3}$ and $\tan\big(\arctan(4/3)\big) = \frac{4}{3}$. The numbers for arctan(1/3)



so $\sec\big(\arctan(1/3)\big) = \frac{\sqrt{10}}{3}$ and $\tan\big(\arctan(1/3)\big) = \frac{1}{3}$. Hence $\displaystyle\int_0^{\ln 4} \frac{e^t dt}{\sqrt{e^{2t}+9}} = \ln\big(\frac{9}{3}\big) - \ln\big(\frac{\sqrt{10}}{3} + \frac{1}{3}\big) = \ln 9 - \ln(1+\sqrt{10}).$

\# 31. $\displaystyle\int_{1/12}^{1/4} \frac{2dt}{\sqrt{t}+4t\sqrt{t}}$. Let $u = 2\sqrt{t}$, $du = \dfrac{dt}{\sqrt{t}}$, so $\displaystyle\int_{1/12}^{1/4}\int_{1/\sqrt{3}}^{1} \frac{2dt}{\sqrt{t}+4t\sqrt{t}} = \int_{1/\sqrt{3}}^{1} \frac{2du}{1+u^2} =$ $\displaystyle 2\arctan u\Big|_{1/\sqrt{3}}^{1} = 2\big(\arctan 1 - \arctan(1/\sqrt{3})\big) = 2\big(\frac{\pi}{4} - \frac{\pi}{6}\big) = \frac{\pi}{6}.$

\# 39. $(x^2+4)y' = 3$, $y(2) = 0$. First separate the differential equation: $dy = 3\dfrac{dx}{x^2+4}$, so $\displaystyle\int dy = 3\int \frac{dx}{x^2+4}$ or $y = \frac{3}{2}\arctan(\frac{x}{2}) + C$. We already have y as a function of x so we can skip this step. To evaluate C, $0 = y(2) = \frac{3}{2}\arctan(\frac{2}{2}) + C = \frac{3}{2}\arctan(1) + C = \dfrac{3\pi}{8} + C$ so $C = -\dfrac{3\pi}{8}$ or $y = \frac{3}{2}\arctan(\frac{x}{2}) - \frac{3\pi}{8}$.

\# 49. $\displaystyle\int \frac{dt}{\sin t - \cos t}$. By the book $z = \tan(t/2)$, $dz = \frac{1}{2}\sec^2(t/2)\, dt = (1+\tan^2(t/2))dt$ or $dt = \dfrac{2dz}{1+z^2}$. Moreover $\cos t = \dfrac{1-z^2}{1+z^2}$; $\sin t = \dfrac{2z}{1+z^2}$. Hence $\displaystyle\int \frac{dt}{\sin t - \cos t} =$ $\displaystyle 2\int \frac{\frac{dz}{1+z^2}}{\frac{2z}{1+z^2} - \frac{1-z^2}{1+z^2}} = 2\int \frac{dz}{-1+2z+z^2}$. Next we need partial fractions. z^2+2z-1 is reducible. It is not easy to see how to factor it, but the quadratic formula locates the roots as $\dfrac{-2\pm\sqrt{2^2-4(1)(-1)}}{2} = \dfrac{-2\pm\sqrt{4+4}}{2} = -1\pm\sqrt{2}$. Hence $z^2+2z-1 = \big(z-(-1+\sqrt{2})\big)\big(z-(-1-\sqrt{2})\big)$. Then $\dfrac{1}{z^2+2z-1} = \dfrac{A}{z-(-1+\sqrt{2})} + \dfrac{B}{z-(-1-\sqrt{2})}$, or $1 = A\big(z-(-1-\sqrt{2})\big) + B\big(z-(-1+\sqrt{2})\big)$. Plug in $z = -1+\sqrt{2}$; $1 = A\big((-1+\sqrt{2})-(-1-\sqrt{2})\big) = A\cdot 2\sqrt{2}$, or $A = \dfrac{1}{2\sqrt{2}}$. Plug in $z = -1-\sqrt{2}$; $1 = B\big((-1-\sqrt{2})-(-1+\sqrt{2})\big)$, or $B = \dfrac{-1}{2\sqrt{2}}$. Hence $\displaystyle\int \frac{dz}{z^2+2z-1} = \frac{1}{2\sqrt{2}}\int\Big(\frac{1}{z-(-1+\sqrt{2})} - \frac{1}{z-(-1-\sqrt{2})}\Big)\, dz =$

$$\frac{1}{2\sqrt{2}}\left(\ln\left|z-(-1+\sqrt{2})\right|-\ln\left|z-(-1-\sqrt{2})\right|\right)+C=\frac{1}{2\sqrt{2}}\ln\left|\frac{z-(-1+\sqrt{2})}{z-(-1-\sqrt{2})}\right|+C.$$

Since $z=\tan(t/2)$, $\displaystyle\int\frac{dt}{\sin t-\cos t}=\frac{1}{\sqrt{2}}\ln\left|\frac{\tan(t/2)+1-\sqrt{2}}{\tan(t/2)+1+\sqrt{2}}\right|+C.$

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