§7.06

1. $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$. This is improper with the only singularity at ∞. Hence $\displaystyle\int_0^\infty \frac{dx}{x^2+1} = \lim_{t\to\infty}\int_0^t \frac{dx}{x^2+1}$. Since $\displaystyle\int_0^t \frac{dx}{x^2+1} = \arctan t - \arctan 0 = \arctan t$ and since $\displaystyle\lim_{t\to\infty}\arctan t = \frac{\pi}{2}$, $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$ converges and its value is $\frac{\pi}{2}$.

5. $\displaystyle\int_{-1}^1 \frac{dx}{x^{2/3}}$. This is improper because the function we are integrating is undefined at 0, a point in the interval of integration. Hence $\displaystyle\int_{-1}^1 \frac{dx}{x^{2/3}} = \int_{-1}^0 \frac{dx}{x^{2/3}} + \int_0^1 \frac{dx}{x^{2/3}} = \lim_{t\to 0^-}\int_{-1}^t \frac{dx}{x^{2/3}} + \lim_{t\to 0^+}\int_t^1 \frac{dx}{x^{2/3}}$. Next do the indefinite integral: $\displaystyle\int \frac{dx}{x^{2/3}} = \int x^{-2/3}dx = \frac{x^{-2/3+1}}{-2/3+1} + C = 3x^{1/3} + C$. Since $\displaystyle\lim_{t\to 0} 3x^{1/3} = 0$, $\displaystyle\lim_{t\to 0^-}\int_{-1}^t \frac{dx}{x^{2/3}} = \lim_{t\to 0^-} 3x^{1/3} - 3{-1}^{1/3} = 0 - 3(-1) = 3$; $\displaystyle\lim_{t\to 0^+}\int_t^1 \frac{dx}{x^{2/3}} = 3(1)^{1/3} - \lim_{t\to 0^+} 3x^{1/3} = 3$ so $\displaystyle\int_{-1}^1 \frac{dx}{x^{2/3}}$ converges and its value is $3+3=6$.

7. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. This integral is improper because the function we are integrating is undefined at 1. Hence $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\arcsin t - \arcsin 0 = \arcsin 1 = \frac{\pi}{2}$. Hence the integral converges and its value is $\frac{\pi}{2}$.

13. $\displaystyle\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2}$. This integral is improper because the end points of the interval of integration are $\pm\infty$. Hence $\displaystyle\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2} = \int_{-\infty}^0 \frac{2x\,dx}{(x^2+1)^2} + \int_0^\infty \frac{2x\,dx}{(x^2+1)^2} = \lim_{t\to-\infty}\int_t^0 \frac{2x\,dx}{(x^2+1)^2} + \lim_{t\to\infty}\int_0^t \frac{2x\,dx}{(x^2+1)^2}$. Do the indefinite integral by substitution: $w = x^2+1$, $dw = 2x\,dx$, so $\displaystyle\int \frac{2x\,dx}{(x^2+1)^2} = \int \frac{dw}{w^2} = \frac{w^{-1}}{-1} + C = \frac{-1}{w} + C = \frac{-1}{x^2+1} + C$. Since $\displaystyle\lim_{x\to\pm\infty}\frac{-1}{x^2+1} = 0$, $\displaystyle\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2} = \left(\frac{-1}{0^2+1} - \lim_{t\to-\infty}\frac{-1}{t^2+1}\right) + \left(\lim_{t\to\infty}\frac{-1}{t^2+1} - \frac{-1}{0^2+1}\right) = (-1-0) - (0-(-1)) = 0$. Hence the integral converges and its value is 0.

17. $\displaystyle\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$. This integral is improper because of the ∞ *and* because the function we are integrating is not defined at 0. Hence $\displaystyle\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{t\to\infty}\int_1^t \frac{dx}{(1+x)\sqrt{x}} +$

$\lim_{t\to 0^+}\int_t^1 \frac{dx}{(1+x)\sqrt{x}}$. Next we do the indefinite integral: substitute $u=\sqrt{x}$, $du=\frac{1}{2}\frac{dx}{\sqrt{x}}$ so $\int \frac{dx}{(1+x)\sqrt{x}} = \int \frac{2du}{1+u^2} = 2\arctan u + C = \arctan(\sqrt{x}) + C$. Hence $\lim_{t\to\infty}\int_1^t \frac{dx}{(1+x)\sqrt{x}} = \lim_{t\to\infty} \arctan(\sqrt{t}) - \arctan(\sqrt{1}) = \frac{\pi}{2} - \arctan 1$ and $\lim_{t\to 0^+}\int_t^1 \frac{dx}{(1+x)\sqrt{x}} = \arctan(\sqrt{1}) - \lim_{t\to 0^+} \arctan(\sqrt{t}) = \arctan(1) - \arctan(0) = \arctan(1) - 0$ so $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ converges and its value is $\frac{\pi}{4}$.

\# 25. $\int_0^1 x\ln x\,dx$. This integral is improper because the function we are integrating is not defined at 0. Hence $\int_0^1 x\ln x\,dx = \lim_{t\to 0^+}\int_t^1 x\ln x\,dx$. Next we do the improper integral $\int x\ln x\,dx$ by parts: $u=\ln x$, $dw = x\,dx$. Then $du=\frac{dx}{x}$ and $w=\frac{x^2}{2}$ so $\int x\ln x\,dx = \frac{x^2\ln x}{2} - \int \frac{x^2}{2}\frac{dx}{x} = \frac{x^2\ln x}{2} - \frac{1}{2}\int x\,dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$. We will have to evaluate $\lim_{t\to 0^+} t^2\ln t$. This limit has the form $0\cdot -\infty$ so we may use l'Hôpital's Rule. $\lim_{t\to 0^+} t^2\ln t = \lim_{t\to 0^+}\frac{\ln t}{1/t^2} = \lim_{t\to 0^+}\frac{1/t}{-2/t^3} = -\lim_{t\to 0^+}\frac{t^2}{2} = 0$. Hence $\int_0^1 x\ln x\,dx = \lim_{t\to 0^+}\left(\frac{x^2\ln x}{2} - \frac{x^2}{4}\right)\Big|_t^1 = \left(\frac{1^2\ln(1)}{2} - \frac{1^2}{4}\right) - \left(\frac{\lim_{t\to 0^+} t^2\ln t}{2} - \frac{(\lim_{t\to 0^+} t)^2}{4}\right) = \frac{-1}{4}$.

\# 37. $\int_0^\pi \frac{\sin\theta}{\sqrt{\pi-\theta}}\,d\theta$ is improper because the function we are integrating is undefined at π. This time we see no way to do the integral, so one of the comparison theorems is required. Between 0 and π, $0\le \sin\theta \le 1$ and $\sqrt{\pi-\theta}\ge 0$ so $0 \le \frac{\sin\theta}{\sqrt{\pi-\theta}} \le \frac{1}{\sqrt{\pi-\theta}}$. We have checked that the first comparison theorem applies. Hence, if $\int_0^\pi \frac{d\theta}{\sqrt{\pi-\theta}}$ converges, so does $\int_0^\pi \frac{\sin\theta}{\sqrt{\pi-\theta}}\,d\theta$. The integral $\int_0^\pi \frac{d\theta}{\sqrt{\pi-\theta}}$ is improper because the function we are integrating is undefined at π. Hence $\int_0^\pi \frac{d\theta}{\sqrt{\pi-\theta}} = \lim_{t\to\pi^-}\int_0^t \frac{d\theta}{\sqrt{\pi-\theta}} = \lim_{t\to\pi^-}\int_0^t (\pi-\theta)^{-1/2}d\theta = \lim_{t\to\pi^-}\frac{-\sqrt{\pi-\theta}}{2}\Big|_0^t = \frac{-\sqrt{0}}{2} - \frac{-\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2}$. Hence $\int_0^\pi \frac{d\theta}{\sqrt{\pi-\theta}}$ converges and therefore so does $\int_0^\pi \frac{\sin\theta}{\sqrt{\pi-\theta}}\,d\theta$.

\# 39. $\int_0^{\ln 2} x^{-2}e^{\frac{-1}{x}}\,dx$ is improper because the function we are integrating is undefined at 0. This integral can actually be done: substitute $u=-x^{-1}$, $du=\frac{dx}{x^2}$ so

$\int_0^{\ln 2} x^{-2} e^{\frac{-1}{x}}\, dx = \int_{-\infty}^{-1/\ln 2} e^u du$. This can be verified as follows: $\int_0^{\ln 2} x^{-2} e^{\frac{-1}{x}}\, dx = \lim_{t\to 0^+} \int_t^{\ln 2} x^{-2} e^{\frac{-1}{x}}\, dx = \lim_{t\to 0^+} \int_{-1/t}^{-1/\ln 2} e^u du = \lim_{t\to -\infty} \int_t^{-1/\ln 2} e^u du = \int_{-\infty}^{-1/\ln 2} e^u du$. Moreover $\lim_{t\to -\infty} \int_t^{-1/\ln 2} e^u du = e^{-1/\ln 2} - \lim_{t\to -\infty} e^t = e^{-1/\ln 2}$. Hence $\int_0^{\ln 2} x^{-2} e^{\frac{-1}{x}}\, dx$ converges (which is all we are asked - but the value is $e^{-1/\ln 2}$).

\# 47. $\int_1^\infty \frac{dx}{x^3+1}$ is improper because we are integrating to ∞. We do not see how to do the integral, so we use a comparison theorem. First note $x^3 < x^3+1$ and $0 \le x^3$ for all x in the interval of integration, $[1,\infty)$. Hence $0 \le \frac{1}{x^3} \le \frac{1}{x^3+1}$ so the first comparison theorem applies and $\int_1^\infty \frac{dx}{x^3+1}$ converges provided $\int_1^\infty \frac{dx}{x^3}$ converges. But $\int_1^\infty \frac{dx}{x^3} = \lim_{t\to\infty} \frac{t^{-2}}{-2} - \frac{1}{-2} = 0 + \frac{1}{2}$. Hence $\int_1^\infty \frac{dx}{x^3}$ converges and therefore so does $\int_1^\infty \frac{dx}{x^3+1}$.

\# 53. $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\, dx$ is improper because we are integrating to ∞. This time we use the limit comparison theorem. First note that $\frac{\sqrt{x+1}}{x^2} \ge 0$: we wish to compare this function to $\frac{\sqrt{x}}{x^2}$. Check that $\lim_{x\to\infty} \frac{\frac{\sqrt{x+1}}{x^2}}{\frac{\sqrt{x}}{x^2}} = \lim_{x\to\infty} \frac{\sqrt{x+1}}{\sqrt{x}} = \sqrt{\lim_{x\to\infty} \frac{x+1}{x}}$. So we need to evaluate $\lim_{x\to\infty} \frac{x+1}{x}$ This has the form $\frac{\infty}{\infty}$ so we may apply l'Hôpital's Rule: $\lim_{x\to\infty} \frac{x+1}{x} = \lim_{x\to\infty} \frac{1}{1} = 1$. Hence the limit comparison theorem may be applied and it says that $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\, dx$ converges if and only if $\int_1^\infty \frac{\sqrt{x}}{x^2}\, dx$ does. But $\int_1^\infty \frac{\sqrt{x}}{x^2}\, dx = \int_1^\infty x^{-3/2}\, dx = \lim_{t\to\infty} \frac{t^{-1/2}}{1/2} - \frac{1}{1/2} = 0 - 2$. This means that $\int_1^\infty \frac{\sqrt{x}}{x^2}\, dx$ converges and hence so does $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\, dx$.

\# 55. $\int_\pi^\infty \frac{2+\cos x}{x}\, dx$ is improper since we are integrating to ∞. We do not know how to do the integral so we use a comparison theorem. Since $\cos x \ge -1$ and since $\frac{1}{x} > 0$ on the interval of integration $[\pi,\infty)$, $0 \le \frac{2-1}{x} \le \frac{2+\cos x}{x}$. Hence the comparison theorem applies so if $\int_\pi^\infty \frac{2-1}{x}\, dx$ diverges, so does $\int_\pi^\infty \frac{2+\cos x}{x}\, dx$. Now $\int_\pi^\infty \frac{2-1}{x} dx = \lim_{t\to\infty} \int_\pi^t \frac{1}{x} dx = \lim_{t\to\infty} \ln t - \ln 1$. But $\lim_{t\to\infty} \ln t = \infty$ so $\int_\pi^\infty \frac{2-1}{x} dx$

diverges and hence so does $\int_{\pi}^{\infty} \frac{2 + \cos x}{x}\, dx$.