§8.01

\# 1. $a_n = \dfrac{1-n}{n^2}$ so $a_1 = \dfrac{1-1}{1^2} = 0$; $a_2 = \dfrac{1-2}{2^2} = -\frac{1}{4}$; $a_3 = \dfrac{1-3}{3^2} = -\frac{2}{9}$; $a_4 = \dfrac{1-4}{4^2} = -\dfrac{3}{16}$.

\# 9. $a_1 = 2$, $a_{n+1} = (-1)^{n+1}a_n/2$. $a_1 = 2$; $a_2 = (-1)^{1+1}\dfrac{2}{2} = 1$;
$a_3 = (-1)^{2+1}\dfrac{1}{2} = -\frac{1}{2}$; $a_4 = (-1)^{3+1}\dfrac{-\frac{1}{2}}{2} = -\frac{1}{4}$; $a_5 = (-1)^{4+1}\dfrac{-\frac{1}{4}}{2} = \frac{1}{8}$;
$a_6 = (-1)^{5+1}\dfrac{\frac{1}{8}}{2} = \frac{1}{16}$; $a_7 = (-1)^{6+1}\dfrac{\frac{1}{16}}{2} = -\frac{1}{32}$; $a_8 = (-1)^{7+1}\dfrac{-\frac{1}{32}}{2} = -\frac{1}{64}$;
$a_9 = (-1)^{8+1}\dfrac{-\frac{1}{64}}{2} = \frac{1}{128}$; $a_{10} = (-1)^{9+1}\dfrac{\frac{1}{128}}{2} = \frac{1}{256}$.

\# 15. The sequence 1, −4, 9, −16, 25, ... *Squares of the positive integers with alternating signs.* $\boxed{a_n = (-1)^{n+1}n^2.\ n \geq 1.}$ Note $a_n = (-1)^n n^2$ has the same verbal discription but does not have the correct first value.

\# 33. $a_n = \dfrac{2^n 3^n}{n!}$. We can rewrite this as $a_n = \dfrac{2\cdot 3}{1}\cdot\dfrac{2\cdot 3}{2}\cdots\dfrac{2\cdot 3}{n}$ so $a_{n+1} = a_n\cdot\dfrac{6}{n+1}$. For $n \geq 6$, $\dfrac{6}{n+1} < 1$ so $a_{n+1} < a_n$. In particular, the sequence is bounded from above by $a_5 = 6\cdot 3\cdot 2\cdot\frac{3}{2}\cdot\frac{6}{5} = \frac{36\cdot 9}{5} = \frac{314}{5} = 62.8$. For $n \geq 6$ the sequence is decreasing, hence *not* non–decreasing. More interestingly, the sequence is bounded below by 0 and since it is decreasing, it does have a limit. Once you know it has a limit, you can compute the limit as follows. Let $\lim\limits_{n\to\infty} a_n = L$. Then $\lim\limits_{n\to\infty} a_{n+1} = L$ also (sequences with the same tails have the same limit) and hence $L = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n\dfrac{6}{n+1} = (\lim\limits_{n\to\infty} a_n)\cdot(\lim\limits_{n\to\infty}\frac{6}{n+1}) = L\cdot 0$ so $L = 0$.

\# 37. $a_n = \dfrac{2^n-1}{2^n}$. There are at least two ways to proceed. One way is to rewrite $a_n = 1 - \dfrac{1}{2^n}$, so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 1 - \dfrac{1}{2^n} = 1 - \lim\limits_{n\to\infty}\dfrac{1}{2^n} = 1 - 0 = 1$ since you are supposed to know $\lim\limits_{n\to\infty}\dfrac{1}{2^n} = 0$. The second way is via L'Hôpital's Rule since $\lim\limits_{n\to\infty} 2^n - 1 = \lim\limits_{n\to\infty} 2^n = \infty$. Hence $\lim\limits_{n\to\infty}\dfrac{2^n-1}{2^n} = \lim\limits_{n\to\infty}\dfrac{(\ln 2)2^n - 0}{(\ln 2)2^n} = \lim\limits_{n\to\infty} 1 = 1$. Hence the sequence converges because we have evaluated the limit. We have used theorems from section 8.2 so these techniques are not what was expected at the time but this is how you should now do these problems. At the time you could have argued as follows. The sequence a_n is increasing which we can show in any of several ways. One way is to write $a_n = 1 - \frac{1}{2^n}$ and remark that $\frac{1}{2^n}$ is decreasing. A second way is to take the derivative of the function $f(x) = \dfrac{2^x-1}{2^x}$; $f'(x) = \dfrac{\big((\ln 2)2^x\big)\cdot 2^x - \big(2^x - 1\big)\cdot\big((\ln 2)2^x\big)}{\big(2^x\big)^2} = \dfrac{(\ln 2)2^x}{2^{2x}} = \dfrac{\ln 2}{2^x} > 0$, so $f(x)$ is an increasing function and hence a_n is an increasing sequence. Since $2^n - 1 < 2^n$, $a_n < 1$ so the sequence is an increasing, bounded sequence and hence converges.

39. $a_n = \big(1 + (1-)^n\big)\left(\frac{n}{n+1}\right)$. Start by noticing that for n odd, $a_n = 0$ since $1 + (-1)^n = 0$. Hence, if $\lim_{n\to\infty} a_n$ exists, it must be 0 since the subsequence $\{a_{2n+1}\}$ converges to 0. (**1**. on page 618). Next consider the subsequence $\{a_{2n}\}$. $a_{2n} = 2 \cdot \frac{2n}{2n+1}$ and $\lim_{n\to\infty} 2 \cdot \dfrac{2n}{2n+1}$ can be computed via L'Hôpital's Rule or by $\lim_{n\to\infty} 2 \cdot \dfrac{2n}{2n+1} = \lim_{n\to\infty} \dfrac{4}{2+\frac{1}{n}} = \dfrac{4}{2+0} = 2$. Since we have two subsequences which converge to two different limits, $\lim_{n\to\infty} a_n$ does not exist. (**2**. on page 618).