§8.02

5. $a_n = \dfrac{1-5n^4}{n^4+8n^3}$. Either use L'Hôpital's Rule four times or rewrite $a_n = \dfrac{\frac{1}{n^4}-5}{1+8\frac{n^3}{n^4}} = \dfrac{\frac{1}{n^4}-5}{1+8\frac{1}{n}} = \dfrac{0-5}{1+8\cdot 0} = -5$. Since we have computed the limit, the sequence converges.

7. $a_n = \dfrac{n^2-2n+1}{n-1}$. You can use L'Hôpital's Rule or else $a_n = \dfrac{n^2-2n+1}{n-1} = \dfrac{(n-1)^2}{n-1} = n-1$ so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n-1 = \infty$ so the sequence diverges. Another way to rewrite a_n is $a_n = n\cdot\dfrac{\frac{n^2}{n^2}-2\cdot\frac{n}{n^2}+\frac{1}{n^2}}{\frac{n}{n}-\frac{1}{n}} = n\cdot\dfrac{1-2\cdot\frac{1}{n}+\frac{1}{n^2}}{1-\frac{1}{n}}$. Now $\lim_{n\to\infty}\dfrac{1-2\cdot\frac{1}{n}+\frac{1}{n^2}}{1-\frac{1}{n}} = 1$ so $\lim_{n\to\infty} a_n = (\lim_{n\to\infty} n)\cdot\left(\lim_{n\to\infty}\dfrac{1-2\cdot\frac{1}{n}+\frac{1}{n^2}}{1-\frac{1}{n}}\right) = \infty\cdot 1 = \infty$.

11. $a_n = \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{n}\right)$. $\lim_{n\to\infty} a_n = \left(\lim_{n\to\infty}\dfrac{n+1}{2n}\right)\left(\lim_{n\to\infty} 1-\dfrac{1}{n}\right) = \frac{1}{2}\cdot 1 = \frac{1}{2}$. This computation shows that the sequence converges to $\frac{1}{2}$.

15. $a_n = \sqrt{\dfrac{2n}{n+1}}$. First do any easier computation: $\lim_{n\to\infty}\dfrac{2n}{n+1} = \lim_{n\to\infty}\dfrac{2}{1+\frac{1}{n}} = 2$. Since the square root function is continuous at 2, the sequence converges and $\lim_{n\to\infty}\sqrt{\dfrac{2n}{n+1}} = \sqrt{\dfrac{2n}{n+1}} = \sqrt{2}$ (see Theorem 4).

27. $a_n = \left(1+\frac{7}{n}\right)^n$. Study $f(x) = \left(1+\frac{7}{x}\right)^x$ for $x > 0$ and compute $\lim_{x\to\infty} f(x)$. This has the form 1^∞ which is indeterminate. Take the log: $\ln f(x) = x\ln\left(1+\frac{7}{x}\right)$. $\lim_{x\to\infty}\ln f(x)$ has the form $\infty\cdot\ln 1 = \infty\cdot 0$, which is again indeterminate. Hence rewrite $x\ln\left(1+\frac{7}{x}\right) = \dfrac{\ln\left(1+\frac{7}{x}\right)}{\frac{1}{x}}$. To compute $\lim_{x\to\infty}\dfrac{\ln\left(1+\frac{7}{x}\right)}{\frac{1}{x}}$ use L'Hôpital's Rule. $\lim_{x\to\infty}\dfrac{\ln\left(1+\frac{7}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\dfrac{\frac{-\frac{7}{x^2}}{1+\frac{7}{x}}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\dfrac{7}{1+\frac{7}{x}} = 7$. Unravelling these calculations, we see $\lim_{x\to\infty}\ln f(x) = 7$, so $\lim_{x\to\infty} f(x) = e^7$ and hence the sequence converges and $\lim_{n\to\infty} a_n = e^7$.

37. $a_n = \frac{n!}{n^n}$. (Hint: compare with $\frac{1}{n}$.) Well, $\frac{n!}{n^n} = \frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n-1}{n}\cdot\frac{n}{n} = \frac{1}{n}\cdot\left(\frac{2}{n}\cdots\frac{n-1}{n}\cdot\frac{n}{n}\right)$. The factor $\frac{2}{n}\cdots\frac{n-1}{n}\cdot\frac{n}{n}$ is a product of positive fractions, each of which is less than 1 so $0 < \frac{2}{n}\cdots\frac{n-1}{n}\cdot\frac{n}{n} < 1$ and hence $0 < a_n < \frac{1}{n}$. Since $\lim_{n\to\infty}\frac{1}{n} = 0$ the Sandwich Theorem shows that the sequence converges and $\lim_{n\to\infty} a_n = 0$.

59. $a_n = n - \sqrt{n^2 - n}$. Study the function $f(x) = x - \sqrt{x^2 - x}$ and in praticular the limit $\lim_{x\to\infty} f(x)$. This has the form $\infty - \infty$ so L'Hôpital's Rule techniques can be used. Write $f(x) = x\cdot\left(1 - \sqrt{1 - \frac{1}{x}}\right) = \dfrac{1 - \sqrt{1 - \frac{1}{x}}}{\frac{1}{x}}$. This last expression has the form $\frac{0}{0}$ as $x \to \infty$ so L'Hôpital's Rule can be applied. $\lim_{x\to\infty} \dfrac{1 - \sqrt{1 - \frac{1}{x}}}{\frac{1}{x}} = \lim_{x\to\infty} \dfrac{0 - \left(\dfrac{-\left(-\frac{1}{x^2}\right)}{2\sqrt{1 - \frac{1}{x}}}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \dfrac{1}{2\sqrt{1 - \frac{1}{x}}} = \dfrac{1}{2\sqrt{1 - 0}} = \dfrac{1}{2}$. Hence the sequence converges and $\lim_{n\to\infty} a_n = \dfrac{1}{2}$.