§8.03

\# 3. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+(-1)^{n-1}\frac{1}{2^{n-1}}+\cdots$ For this series find the nth partial sum and compute the limit to see if the series converges. Now $s_n = 1-\frac{1}{2}+\cdots+(-1)^{n-1}\frac{1}{2^{n-1}}$ since we are supposed to sum n of the terms. By the usual formula for a geometric series, $s_n = \dfrac{1-(-\frac{1}{2})^n}{1-(-\frac{1}{2})} = \frac{2}{3}\cdot\left(1-(-\frac{1}{2})^n\right)$. Hence $\lim\limits_{n\to\infty} s_n = \dfrac{2}{3}\cdot 1$ so $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+(-1)^{n-1}\frac{1}{2^{n-1}}+\cdots$ converges and is equal to $\frac{2}{3}$.

\# 7. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n} = 1-\frac{1}{4}+\frac{1}{16}-\frac{1}{32}+\cdots$. This is a geometric series with $a=1$ and $r=-\frac{1}{4}$. Since $|r|<1$ this series converges and $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n} = \frac{1}{1-\left(-\frac{1}{4}\right)} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$.

\# 11. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right) = \sum_{n=0}^{\infty}\frac{5}{2^n}+\sum_{n=0}^{\infty}\frac{1}{3^n}$. First $\displaystyle\sum_{n=0}^{\infty}\frac{5}{2^n}$ is a geometric series with $a=5$ and $r=\frac{1}{2}$. Since $|r|<1$, the series converges and $\displaystyle\sum_{n=0}^{\infty}\frac{5}{2^n} = \frac{5}{1-\frac{1}{2}} = 10$. Next $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3^n}$ is a geometric series with $a=1$ and $r=\frac{1}{3}$. Since $|r|<1$ this series converges and $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$. Hence $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)$ converges to $10+\frac{3}{2}=\frac{23}{2}$.

\# 17. $\displaystyle\sum_{n=1}^{\infty}\frac{40n}{(2n-1)^2(2n+1)^2}$. First use partial fractions: $\dfrac{40n}{(2n-1)^2(2n+1)^2} = \dfrac{A}{2n-1}+\dfrac{B}{(2n-1)^2}+\dfrac{C}{2n+1}+\dfrac{D}{(2n+1)^2}$ or $40n = A(2n-1)(2n+1)^2+B(2n+1)^2+C(2n-1)^2(2n+1)+D(2n-1)^2$. Plug in $n=\frac{1}{2}$; $20 = A\cdot 0+B(2)^2+C\cdot 0+D\cdot 0$ or $B=5$. Plug in $n=-\frac{1}{2}$ to see $D=-5$. This means $40n = A(2n-1)(2n+1)^2+C(2n-1)^2(2n+1)+5\left((2n+1)^2-(2n-1)^2\right) = A(2n-1)(2n+1)^2+C(2n-1)^2(2n+1)+40n$, so $A(2n-1)(2n+1)^2+C(2n-1)^2(2n+1)=0$, which can only happen if $A=C=0$. Hence $\dfrac{40n}{(2n-1)^2(2n+1)^2} = \dfrac{5}{(2n-1)^2}-\dfrac{5}{(2n+1)^2}$ so the series is a telescoping one. Hence $\displaystyle\sum_{n=1}^{\infty}\frac{40n}{(2n-1)^2(2n+1)^2} = \frac{5}{1^2}-\lim_{n\to\infty}\frac{5}{(2n+1)^2} = 5$.

\# 23. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$. This is a geometric series with $a=1$ and $r=\dfrac{1}{\sqrt{2}}$. Since $\sqrt{2}>1$, $\dfrac{1}{\sqrt{2}}<1$ so this series converges. From the formula $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}(\sqrt{2}+1)}{2-1} = 2+\sqrt{2}$.

\# 29. $\displaystyle\sum_{n=0}^{\infty}e^{-2n}$ is a geometric series with $a=1$ and $r=e^{-2}$. Since $e>1$, $e^2>1$ and $|r|<1$.

Hence the series converges and $\displaystyle\sum_{n=0}^{\infty} e^{-2n} = \frac{1}{1-e^{-2}} = \frac{e^2}{e^2-1}$.

35. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{1000^n}$. You should always wonder about $\lim_{n\to\infty} a_n$. In this case, $\frac{n!}{1000^n} = \frac{1}{1000} \cdot \frac{2}{1000} \cdots \frac{n-1}{1000} \cdot \frac{n}{1000}$. Hence $a_{n+1} = a_n \cdot \frac{n+1}{1000}$. Hence the sequence a_n is increasing for $n \geq 1000$. Hence either $\lim_{n\to\infty} a_n = L$ if the sequence a_n is bounded or else $\lim_{n\to\infty} a_n = \infty$. As usual, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ and $\lim_{n\to\infty} a_{n+1} = (\lim_{n\to\infty} a_n) \cdot \left(\lim_{n\to\infty} \frac{n+1}{1000}\right)$. Since $\displaystyle\lim_{n\to\infty} \frac{n+1}{1000} = \infty$, the only choice for $\lim_{n\to\infty} a_n$ is ∞. Hence the series diverges.

43. $\displaystyle\sum_{n=0}^{\infty} 3\Big(\frac{x-1}{2}\Big)^n$ is a geometric series with $a = 3$ and $r = \frac{x-1}{2}$. The series converges whenever $|r| < 1$ or $\left|\frac{x-1}{2}\right| < 1$. Hence $|x-1| < 2$ so $-2 < x-1 < 2$ or $-1 < x < 3$. Hence the series converges for any number x, $-1 < x < 3$. We weren't asked, but we can sum this series, $\displaystyle\sum_{n=0}^{\infty} 3\Big(\frac{x-1}{2}\Big)^n = \frac{3}{1-\frac{x-1}{2}} = \frac{6}{2-(x-1)} = \frac{6}{3-x}$.

45. $\displaystyle\sum_{n=0}^{\infty} 2^n x^n$ is a geometric series with $a = 1$ and $r = 2x$. Hence the series converges precisely when $|2x| < 1$ or $-\frac{1}{2} < x < \frac{1}{2}$. For such x, $\displaystyle\sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}$.

57. $1.24\overline{123} = 1.24 + \displaystyle\sum_{n=1}^{\infty} \frac{123}{100{\cdot}1000^n}$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{123}{100{\cdot}1000^n}$ is a geometric series with $a = \frac{123}{100\cdot 1000}$ and $r = \frac{1}{1000}$. Hence $\displaystyle\sum_{n=1}^{\infty} \frac{123}{100{\cdot}1000^n} = \frac{\frac{123}{100\cdot 1000}}{1-\frac{1}{1000}} = \frac{123}{100000-100} = \frac{123}{99900}$, so $1.24\overline{123} = \frac{124}{100} + \frac{123}{99900} = \frac{124*999+123}{99900} = \frac{124000-1}{99900} = \frac{123999}{99900} = \frac{41333}{33300}$.

59. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=-2}^{\infty} \frac{1}{(n+4)(n+5)}$; (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+3)}$; (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=5}^{\infty} \frac{1}{(n-3)(n-2)}$.