§8.04

\# 3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$. First compute $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$. Hence the series diverges.

\# 5. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = 3\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$. Now $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a p–series with $p = \frac{1}{2}$ and since $\frac{1}{2} < 1$, it diverges. Hence the original series diverges.

\# 11. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ is a geometric series with $r = \frac{2}{3}$. Since $\frac{2}{3} < 1$, this series converges.

\# 21. $\displaystyle\sum_{n=3}^{\infty} \frac{1/n}{(\ln n)\sqrt{\ln^2 n - 1}}$. We want to use the Integral Test with $f(x) = \dfrac{1}{x(\ln x)\sqrt{\ln^2 x - 1}}$. First we need to check that $f(x)$ is non–negative and decreasing. Equivalently, we can check that $x(\ln x)\sqrt{\ln^2 x - 1}$ is non–negative and increasing. Both x and $\ln x$ are positive and increasing for $x > 1$; it follows that $\ln^2 x - 1$ is also increasing and hence $\sqrt{\ln^2 x - 1}$ is also increasing if $\ln x > 1$ or $x > e$. Therefore, for $x \geq 3$, the product of $x(\ln x)\sqrt{\ln^x -1}$ is positive and increasing. Hence the series converges or diverges provided $\displaystyle\int_3^{\infty} \frac{1}{x(\ln x)\sqrt{\ln^2 x - 1}}\, dx$ converges or diverges. Rather than do this integral directly, we use the Limit Comparison Test on the function $g(x) = \dfrac{1}{x\, ln^2 x}$. First compute $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{x(\ln x)\sqrt{\ln^2 x - 1}}}{\frac{1}{x\, ln^2 x}} = \lim_{x\to\infty} \frac{\ln x}{\sqrt{\ln^2 x - 1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1 - \frac{1}{\ln^2 x}}} = 1$. The integral $\displaystyle\int \frac{dx}{x \ln^2 x}$ can be done by the susbtitution $u = \ln x$, $du = \frac{dx}{x}$ so $\displaystyle\int \frac{dx}{x\ln^2 x} = \int \frac{du}{u^2} = -u^{-1} + C = -\frac{1}{\ln x} + C$ so $\displaystyle\int_3^{\infty} \frac{dx}{x\ln^2 x} = \lim_{t\to\infty} -\frac{1}{\ln t} - (-\frac{1}{\ln 3}) = \frac{1}{\ln 3}$ so the improper integral converges. It follows from the Limit Comparison Test that $\displaystyle\int_3^{\infty} \frac{dx}{x(\ln x)\sqrt{\ln^2 x - 1}}$ converges and therefore the series converges.

The original integral can also be done by the same substitution, $u = \ln x$: the integral becomes $\displaystyle\int \frac{1}{x(\ln x)\sqrt{\ln^2 x - 1}}\, dx = \int \frac{1}{u\sqrt{u^2 - 1}}\, du = \text{arcsec} u + C = \text{arcsec}(\ln x) + C$. Hence $\displaystyle\int_3^{\infty} \frac{1}{x(\ln x)\sqrt{\ln^2 x - 1}} dx$ converges if and only if $\displaystyle\lim_{x\to\infty} \text{arcsec}(\ln x)$ exits and is finite. Since $\displaystyle\lim_{x\to\infty} \ln x = \infty$, $\displaystyle\lim_{x\to\infty} \text{arcsec}(\ln x) = \lim_{t\to\infty} \text{arcsec}(t) = \frac{\pi}{2}$. Hence $\displaystyle\int_3^{\infty} \frac{1}{x(\ln x)\sqrt{\ln^2 x - 1}} dx$ converges, as does the series.

33. $a_1 = 1$, $a_{n+1} = \frac{1+\ln n}{n} a_n$: does $\sum_{n=1}^{\infty} a_n$ converge or diverge? The Ratio Test appears the way to go since from the definition, $\frac{a_{n+1}}{a_n} = \frac{1+\ln n}{n}$. Hence $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{\ln n}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ so $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$ and the series converges.

39. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$. We rewrite the nth term of the series so as to get an idea of how fast it is going to 0. First write $\frac{(n!)^n}{(n^n)^2} = \left(\frac{n!}{n^2}\right)^n$. Now apply the Root Test: $\lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{(n^n)^2}} = \lim_{n\to\infty} \frac{n!}{n^2}$. Then note that $\frac{n!}{n^2} = (n-2)! \cdot \frac{n-1}{n} = (n-3)! \cdot \frac{(n-2)(n-1)}{n}$. If $n \geq 4$, $\frac{(n-2)(n-1)}{n} > 1$ and $(n-3)! \geq n-3$ so $\frac{n!}{n^2} > n-3$ and $\lim_{n\to\infty} \frac{n!}{n^2} = \infty$. In particular, the series diverges.

41. $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$. As in #39, let us start by studying $\frac{n^n}{2^{n^2}}$. $2^{n^2} = 2^{n \cdot n} = \left(2^n\right)^n$. Hence $\frac{n^n}{2^{n^2}} = \left(\frac{n}{2^n}\right)^n$. The Root Test again appears the way to go: $\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{2^{n^2}}} = \lim_{n\to\infty} \frac{n}{2^n} = 0$ where this last limit can be evaluated by L'Hôpital's Rule if you don't remember that an exponential function with a base greater than 1 increases faster than any polynomial. Hence the series converges.