§8.05

These are to be done using one of the two comparison tests. To use these tests you need to know examples. The two families that you know are

- The geometric series $\sum^{\infty} r^n$ converges for $|r| < 1$ and diverges otherwise
- The p–series, $\sum^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges otherwise.

\# 1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$. Compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$: $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n} + \sqrt[3]{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n} + \sqrt[3]{n}} = \lim_{n\to\infty} \frac{1}{2 + \frac{\sqrt[3]{n}}{\sqrt{n}}} =$ $\displaystyle\lim_{n\to\infty} \frac{1}{2 + \frac{1}{n^{\frac{1}{6}}}} = \frac{1}{2} \neq 0$. Hence the two series diverge or converge together and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p–series with $p = \frac{1}{2} < 1$ and so both series diverge.

\# 3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$. Compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$. This is a geometric series with $r = \frac{1}{2} < 1$ and so converges. Since $0 \leq \sin^2 n < 1$, the first comparison test shows that $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ comverges.

\# 5. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-1}$. Since $\displaystyle\lim_{n\to\infty} \frac{2n}{3n-1} = \frac{2}{3} \neq 0$ the series diverges since the limit of the terms is not 0.

\# 7. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$. Compare to $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ which is a geometric series with $r = \frac{1}{3} < 1$ and hence converges. Since $0 < \frac{n}{3n+1} = \frac{1}{3 + \frac{1}{n}} < \frac{1}{3}$, the first comparison test shows that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$ converges.

\# 21. $\displaystyle\sum_{n=1}^{\infty} \frac{1-n}{n2^n}$. Note $\displaystyle\sum_{n=1}^{\infty} \frac{1-n}{n2^n} = -\sum_{n=1}^{\infty} \frac{n-1}{n2^n}$ so we can determine the behavior of $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n2^n}$ which has non–negative terms so our theorems apply. Compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ which is a geometric series with $r = \frac{1}{2} < 1$ and hence converges. Compute $\displaystyle\lim_{n\to\infty} \frac{\frac{n-1}{n2^n}}{\frac{1}{2^n}} =$ $\displaystyle\lim_{n\to\infty} \frac{n-1}{n} = 1 \neq 0$ so by the limit comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n2^n}$ converges and hence so does $\displaystyle\sum_{n=1}^{\infty} \frac{1-n}{n2^n}$.

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{n-1}+1}$. Compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ which is a geometric series with $r = \frac{1}{3} < 1$ and hence converges,. Compute $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{3^{n-1}+1}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^{n-1}+1} = \lim_{n\to\infty} \frac{3}{1+\frac{1}{3^{n-1}}} = 3$ so by the limit comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{n-1}+1}$ converges.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$. Compare to $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a p–series with $p = 2 > 1$ and hence converges. Compute $\displaystyle\lim_{n\to\infty} \frac{\frac{10n+1}{n(n+1)(n+2)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{10n^3+n^2}{n(n+1)(n+2)} = 10 \neq 0$. Hence $\displaystyle\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$ converges by the limit comparison test.

29. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.1}}$. To understand the behavior of the terms, $\frac{\arctan n}{n^{1.1}}$, we recall the behavior of $\arctan x$ as $x \to \infty$: $\displaystyle\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$. Since $\frac{d \arctan x}{dx} = \frac{1}{1+x^2} > 0$ it follows that $\arctan n < \frac{\pi}{2}$. Compare to $\displaystyle\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^{1.1}} = \frac{\pi}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$. Now $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ is a p–series with $p = 1.1 > 1$ and hence convergent. It follows that $\displaystyle\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^{1.1}}$ converges and since $0 < \frac{\arctan n}{n^{1.1}} < \frac{\frac{\pi}{2}}{n^{1.1}}$, it follows from the first comparison test that $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.1}}$ converges.

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$. Again we need to understand the individual terms better. A useful formula is $1+2+\cdots+n = \frac{n(n+1)}{2}$. Hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n} = \sum_{n=1}^{\infty} \frac{2}{n(n+1)}$. This can be compared to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a p–series with $p = 2 > 1$ and hence converges. Compute $\displaystyle\lim_{n\to\infty} \frac{\frac{2}{n(n+1)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^2}{n(n+1)} = 2 \neq 0$. It follows from the limit comparison that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$ converges.