§8.06

1. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{\sqrt{2}}}{2^n}$. To do the ratio test we need to compute $\displaystyle\lim_{n\to\infty}\frac{\frac{(n+1)^{\sqrt{2}}}{2^{n+1}}}{\frac{n^{\sqrt{2}}}{2^n}} = \lim_{n\to\infty}\tfrac{1}{2}\left(\frac{n+1}{n}\right)^{\sqrt{2}} = \tfrac{1}{2}\left(\lim_{n\to\infty}\frac{n+1}{n}\right)^{\sqrt{2}} = \tfrac{1}{2} < 1$ so by the ratio test, this series converges. The root test can also be made to work: $\displaystyle\lim_{n\to\infty}\sqrt[n]{\frac{n^{\sqrt{2}}}{2^n}} = \frac{\left(\lim_{n\to\infty}\sqrt[n]{n}\right)^{\sqrt{2}}}{2} = \tfrac{1}{2}$.

3. $\displaystyle\sum_{n=1}^{\infty} n!e^{-n}$. The ratio test works well here. Compute $\displaystyle\lim_{n\to\infty}\frac{(n+1)!\,e^{-(n+1)}}{n!\,e^{-n}} = \lim_{n\to\infty}\frac{n+1}{e} = \infty > 1$ so the ratio test shows that this series diverges. The root test is harder to get to work. The difficulty can be reduced to the computation $\displaystyle\lim_{n\to\infty}\sqrt[n]{n!} = \infty$ but this is difficult to show directly.

5. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{10}}{10^n}$. The ratio test requires the computation $\displaystyle\lim_{n\to\infty}\frac{\frac{(n+1)^{10}}{10^{n+1}}}{\frac{n^{10}}{10^n}} = \frac{1}{10}\left(\lim_{n\to\infty}\frac{n+1}{n}\right)^{10} = \frac{1}{10} < 1$ so the ratio test shows that the series converges. The root test works a follows: $\displaystyle\lim_{n\to\infty}\sqrt[n]{\frac{n^{10}}{10^n}} = \frac{\left(\lim_{n\to\infty}\sqrt[n]{n}\right)^{10}}{10} = \frac{1}{10}$.

11. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^3}$. The ratio test requires the calculation $\displaystyle\lim_{n\to\infty}\frac{\frac{\ln(n+1)}{(n+1)^3}}{\frac{\ln n}{n^3}} = \lim_{n\to\infty}\frac{n^3\ln(n+1)}{(n+1)^3\ln n} = \left(\lim_{n\to\infty}\frac{n}{n+1}\right)^3\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} = 1$ since one can use L'Hôpital's Rule to show $\displaystyle{}_{x\to\infty}\frac{\ln(x+1)}{\ln x} = 1$. The root test also gives 1: the hard part is to show $\displaystyle\lim_{n\to\infty}\sqrt[n]{\ln n} = 1$. In any case, both tests are inconclusive. We solve the problem by appealing to one of the earlier comparison tests: we compare our series to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$. This is a p–series with $p = 2 > 1$ so this second series converges. Compute $\displaystyle\lim_{n\to\infty}\frac{\frac{\ln n}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\ln n}{n} = 0$ so $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^3}$ converges since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ does.

19. $\displaystyle\sum_{n=1}^{\infty}\frac{(n+3)!}{3!\,n!\,3^n}$. The calculation for the ratio test is $\displaystyle\lim_{n\to\infty}\frac{\frac{(n+4)!}{3!\,(n+1)!\,3^{n+1}}}{\frac{(n+3)!}{3!\,n!\,3^n}} = \lim_{n\to\infty}\frac{n+4}{3(n+1)} =$

$\frac{1}{3} < 1$. Hence the series converges.

25. $\sum_{n=1}^{\infty} \frac{n!\ \ln n}{n(n+2)!}$. Use the ratio test again: $\lim_{n\to\infty} \frac{\frac{(n+1)!\ \ln(n+1)}{(n+1)(n+3)!}}{\frac{n!\ \ln n}{n(n+2)!}} = \lim_{n\to\infty} \frac{n\ \ln(n+1)}{(n+3)\ln n} = \lim_{n\to\infty} \frac{n}{n+3} \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = 1$ so the ratio test is inconclusive. To continue, rewrite the series as follows $\sum_{n=1}^{\infty} \frac{n!\ \ln n}{n(n+2)!} = \sum_{n=1}^{\infty} \frac{\ln n}{n(n+1)(n+2)}$. Compare this to the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ from #11. This series converges by #11. Compute the limit $\lim_{n\to\infty} \frac{\frac{\ln n}{n(n+1)(n+2)}}{\frac{\ln n}{n^3}} = \lim_{n\to\infty} \frac{n(n+1)(n+2)}{n^3} = 1 \neq 0$. Hence $\sum_{n=1}^{\infty} \frac{n!\ \ln n}{n(n+2)!}$ converges.

27. $a_1 = 2$, $a_{n+1} = \frac{1+\sin n}{n}\, a_n$. This recursive definition is ideal for the ratio test since $\frac{a_{n+1}}{a_n} = \frac{1+\sin n}{n}$, so $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1+\sin n}{n}$. Since $0 < \frac{1+\sin n}{n} < \frac{2}{n}$, the Sandwich Theorem shows $\lim_{n\to\infty} \frac{1+\sin n}{n} = 0 < 1$. Hence the series converges by the ratio test.

31. $a_1 = 2$, $a_{n+1} = \frac{2}{n}\, a_n$. The same idea works well here. $\frac{a_{n+1}}{a_n} = \frac{2}{n}$ so $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$, so again the series converges.