3. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$. To use any of our tests, it is important to understand how the terms of the series approach 0. As a first step, consider $\lim_{n\to\infty}\left(\frac{n}{10}\right)^n = \infty$. This means that the terms are not approaching 0 at all and so the series diverges.

5. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n}$. This time the terms do go to 0. Since $\ln x$ is an increasing function, $\frac{1}{\ln n}$ is decreasing, so by the Alternating Series Test this series converges. The series $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges since $\ln n < n$, so $\frac{1}{n} < \frac{1}{\ln n}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges since it is a p–series with $p = 1$. Hence this is a conditionally convergent series.

9. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$. Check $\lim_{n\to\infty}\frac{\frac{\sqrt{n}+1}{n+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{n+\sqrt{n}}{n+1} = \lim_{n\to\infty}\frac{1+\frac{1}{\sqrt{n}}}{1+\frac{1}{n}} = 1$. Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a p–series with $p = \frac{1}{2} < 1$, it diverges and hence so does $\sum_{n=1}^{\infty}\frac{\sqrt{n}+1}{n+1}$. Check that $\lim_{n\to\infty}\frac{\sqrt{n}+1}{n+1} = \lim_{n\to\infty}\frac{1+\frac{1}{\sqrt{n}}}{\sqrt{n}+\frac{1}{\sqrt{n}}} = 0$. To apply the Alternating Series Test, it remains to check that $\frac{\sqrt{n}+1}{n+1}$ is decreasing. To show this, study the function $f(x) = \frac{\sqrt{x}+1}{x+1}$. Compute $f'(x) = \frac{\frac{d\sqrt{x}+1}{dx}(x+1) - (\sqrt{x}+1)\frac{dx+1}{dx}}{(x+1)^2} = \frac{\frac{1}{2\sqrt{x}}(x+1) - (\sqrt{x}+1)}{(x+1)^2} = \frac{\frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2} - 1}{(x+1)^2}$. For $x \geq 1$, $\sqrt{x} \geq 1$ so $\frac{1}{2\sqrt{x}} \leq \frac{1}{2}$ so $\frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2} - 1 < 0$ and hence $f'(x) < 0$ for all $x \geq 1$. Hence the terms $\frac{\sqrt{n}+1}{n+1}$ are decreasing and the Alternating Series Test applies to show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$ converges. Since we have already shown $\sum_{n=1}^{\infty}\frac{\sqrt{n}+1}{n+1}$ diverges, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$ is a conditionally convergent series.

13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$. To use the Alternating Series Test first check that $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ and that the terms $\frac{1}{\sqrt{n}}$ are decreasing (since $\sqrt{x}$ is an increasing function). These two calculations show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ converges. Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a p–series with

$p = \frac{1}{2} < 1$, this series diverges, so $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ is a conditionally convergent series.

15. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$. Compare $\dfrac{n}{n^3+1}$ to $\dfrac{1}{n^2}$: $\displaystyle\lim_{n\to\infty}\frac{\frac{n}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^3}{n^3+1} = 1$. Hence $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$ absolutely converges because $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p–series with $p = 2 > 1$ and so converges.

17. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n+3}$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+3} = \sum_{n=4}^{\infty}\frac{1}{n}$ which is a p–series with $p = 1$ and so diverges. Since $\displaystyle\lim_{n\to\infty}\frac{1}{n+3} = 0$ and since $n+3$ is increasing (so $\dfrac{1}{n+3}$ is decreasing), the Alternating Series Test shows $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n+3}$ converges. Therefore $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n+3}$ conditionally converges.

19. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$ This time $\displaystyle\lim_{n\to\infty}\frac{3+n}{5+n} = \frac{3}{5} \neq 0$ so the terms of this series do not go to 0 so the series diverges.

31. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+2n+1}$. Compute $\displaystyle\lim_{n\to\infty}\frac{\frac{1}{n^2+2n+1}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{n^2+2n+1} = 1$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p–series with $p = 2 > 1$, it converges and hence $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+2n+1}$ converges absolutely. One can also proceed as follows. Since $n^2+2n+1 = (n+1)^2$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+2n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^2} = \sum_{n=2}^{\infty}\frac{(-1)^{n}}{n^2}$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p–series with $p = 2 > 1$, it converges, so $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+2n+1}$ converges absolutely.

35. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}(n+1)^n}{(2n)^n}$. Compute $\displaystyle\lim_{n\to\infty}\frac{\frac{(n+1)^n}{(2n)^n}}{\frac{1}{2^n}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ as we saw back in the L'Hôpital's Rule section. The series $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^n}$ is a geometric series with $r = \frac{1}{2}$ and so converges. Hence $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}(n+1)^n}{(2n)^n}$ converges absolutely.

45. Estimate the difference between $\ln 2$ and the first four terms of the series $\ln 2 =$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}.$$

The book checks that the Alternating Series Test applies so $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}-\sum_{n=1}^{4}(-1)^{n+1}\frac{1}{n} = \sum_{n=5}^{\infty}(-1)^{n+1}\frac{1}{n}$. The Alternating Series Test says $0 < \sum_{n=5}^{\infty}(-1)^{n+1}\frac{1}{n} < \frac{1}{5}$ so

$$\frac{7}{12} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} < \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} < 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$$