§8.08

3. $\sum_{n=0}^{\infty}(-1)^n(4x+1)^n$ is not in the usual form, we need $(x-b)^n$, so we factor out a 4^n: $\sum_{n=0}^{\infty}(-1)^n(4x+1)^n = \sum_{n=0}^{\infty}(-1)^n 4^n(x+\frac{1}{4})^n = \sum_{n=0}^{\infty}(-4)^n\big(x-(-\frac{1}{4})\big)^n$. To calculate the radius of convergence, use either the radius or the root test. For the ratio test, compute $\lim_{n\to\infty}\frac{|(-4)^{n+1}|R^{n+1}}{|(-4)^n|R^n} = 4R$ and $4R = 1$ or $R = \frac{1}{4}$. For the root test, compute $\lim_{n\to\infty}\sqrt[n]{|(-4)^n|R^n} = 4R$ so again $R = \frac{1}{4}$. At the two endpoints, we need to evaluate the the convergence properties of $\sum_{n=0}^{\infty}(-4)^n(\frac{1}{4})^n$ and $\sum_{n=0}^{\infty}(-4)^n(-\frac{1}{4})^n$.

- $\sum_{n=0}^{\infty}(-4)^n(\frac{1}{4})^n = \sum_{n=0}^{\infty}(-1)^n$ diverges since $\lim_{n\to\infty}(-1)^n$ does not exists and henceis not 0
- $\sum_{n=0}^{\infty}(-4)^n(-\frac{1}{4})^n = \sum_{n=0}^{\infty}1$ diverges since $\lim_{n\to\infty} 1 \neq 0$

Putting this all together, the series converges absolutely for x in the interval $(-\frac{1}{2}, 0)$ and diverges for all x not in this open interval.

7. $\sum_{n=0}^{\infty}\frac{n\,x^n}{n+2}$ To compute the radius of convergence, use the ratio test: $\lim_{n\to\infty}\frac{\frac{n+1}{n+3}R^{n+1}}{\frac{n}{n+2}R^n} = \lim_{n\to\infty}\frac{(n+1)(n+2)R^{n+1}}{n(n+3)R^n} = \lim_{n\to\infty}\frac{(n+1)(n+2)}{n(n+3)}R = \lim_{n\to\infty}\frac{(1+\frac{1}{n})(1+\frac{2}{n})}{1+\frac{3}{n}}R = R$ so the radius of convergence is 1. At the end points, we must study the series $\sum_{n=0}^{\infty}\frac{n}{n+2}$ and $\sum_{n=0}^{\infty}\frac{n\,(-1)^n}{n+2}$. Since $\lim_{n\to\infty}\frac{n}{n+2} = 1$ both series diverge since the limit of the terms of either sequence do not go to 0. Hence the series converges absolutely for all x in the interval $(-1, 1)$ and it diverges for all other x.

13. $\sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!}$. Use the ratio test and compute $\lim_{n\to\infty}\frac{\frac{1}{(n+1)!}R^{n+1}}{\frac{1}{n!}R^n} = \lim_{n\to\infty}\frac{R}{n} = 0$ for any R so the radius of convergence is infinity. Hence the series converges absolutely everywhere.

19. $\sum_{n=0}^{\infty}\frac{\sqrt{n}\,x^n}{3^n}$ Use the ratio test and compute $\lim_{n\to\infty}\frac{\frac{\sqrt{n+1}}{3^{n+1}}R^{n+1}}{\frac{\sqrt{n}}{3^n}R^n} = \lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n}}\frac{R}{3} = \lim_{n\to\infty}\sqrt{\frac{n+1}{n}}\frac{R}{3} = \frac{R}{3}$ so the radius of convergence is 3. At the two end points we need

to study the two series $\sum_{n=0}^{\infty} \frac{\sqrt{n}\, 3^n}{3^n} = \sum_{n=0}^{\infty} \sqrt{n}$ and $\sum_{n=0}^{\infty} \frac{\sqrt{n}\,(-3)^n}{3^n} = \sum_{n=0}^{\infty} \sqrt{n}(-1)^n$. Since $\lim_{n\to\infty} \sqrt{n} = \infty$, both the above series diverge since the limit of the terms in the series do not go to 0. Hence the series converges absolutely for x in $(-3, 3)$ and diverges for all other x.

\# 29. $\sum_{n=1}^{\infty} \frac{(4x-5)^{2n+1}}{n^{3/2}}$. As in #3, we first need to rewrite the series in standard form: $\sum_{n=1}^{\infty} \frac{(4x-5)^{2n+1}}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{4^{2n+1}}{n^{3/2}} \left(x - \frac{5}{4}\right)^{2n+1}$. Compute the radius of convergence using the root test (the ratio test can also be made to work). $\lim_{n\to\infty} \sqrt[n]{\frac{4^{2n+1}}{n^{3/2}} R^{2n+1}} = \lim_{n\to\infty} \frac{4^{2+\frac{1}{n}}}{\left(n^{\frac{1}{n}}\right)^{3/2}} R^{2+\frac{1}{n}}$. Since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, $\lim_{n\to\infty} \sqrt[n]{\frac{4^{2n+1}}{n^{3/2}} R^{2n+1}} = 4^2 R^2$ so $R^2 = \frac{1}{4^2}$ so $R = \frac{1}{4}$. Understanding the endpoints involves understanding the series $\sum_{n=1}^{\infty} \frac{4^{2n+1}}{n^{3/2}} \left(\frac{1}{4}\right)^{2n+1} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{4^{2n+1}}{n^{3/2}} \left(-\frac{1}{4}\right)^{2n+1} = \sum_{n=1}^{\infty} \frac{-1}{n^{3/2}}$. Since one of the series is the negative of the other, either both converge absolutely or both diverge. Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a p–series with $p = \frac{3}{2}$ and since $\frac{3}{2} > 1$, this p–series converges. Hence the series converges absolutely for all x in the interval $[1, \frac{3}{2}]$.

\# 33. $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$. To compute the radius of convergence, use the root test $\lim_{n\to\infty} \sqrt[n]{\frac{R^{2n}}{4^n}} = \frac{R^2}{4}$ so the radius of convergence is 2. The two endpoint series are $\sum_{n=0}^{\infty} \frac{(2)^{2n}}{4^n} = \sum_{n=0}^{\infty} 1$ and $\sum_{n=0}^{\infty} \frac{(-2)^{2n}}{4^n} = \sum_{n=0}^{\infty} 1$ so both series diverge as the terms do not go to 0. To actually sum the series note that it is a geometric series $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n} = \sum_{n=0}^{\infty} \left(\frac{(x-1)^2}{4}\right)^n = \frac{1}{1 - \frac{(x-1)^2}{4}} = \frac{4}{4-(x^2-2x+1)} = \frac{4}{3+2x-x^2}$

\# 41. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$

a) Since $\frac{d \sin x}{dx} = \cos x$ and we can differentiate the series term by term, so $\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!} - \frac{11x^{10}}{11!} + \cdots$ or $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$

b) $\sin(2x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \frac{(2x)^9}{9!} - \frac{(2x)^{11}}{11!} + \cdots$ or $\sin(2x) = 2x - \frac{8x^3}{3!} +$

$\frac{32x^5}{5!} - \frac{128x^7}{7!} + \frac{512x^9}{9!} - \frac{2048x^{11}}{11!} + \cdots$

c) Since $\sin(2x) = 2\cos x \sin x$ we can use the product theorem to compute

$$\begin{aligned}
\cos x \sin x &= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots) \cdot (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots) \\
&= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \frac{x^9}{8!} - \frac{x^{11}}{10!} + \cdots \\
&\qquad - \frac{x^3}{3!} + \frac{x^5}{3! \cdot 2!} - \frac{x^7}{3! \cdot 4!} + \frac{x^9}{3! \cdot 6!} - \frac{x^{11}}{3! \cdot 8!} + \cdots \\
&\qquad \frac{x^5}{5!} - \frac{x^7}{5! \cdot 2!} + \frac{x^9}{5! \cdot 4!} - \frac{x^{11}}{5! \cdot 6!} + \cdots \\
&\qquad - \frac{x^7}{7!} + \frac{x^9}{7! \cdot 2!} - \frac{x^{11}}{7! \cdot 4!} + \cdots \\
&\qquad \frac{x^9}{9!} - \frac{x^{11}}{9! \cdot 2!} + \cdots \\
&\qquad - \frac{x^{11}}{11!} - \cdots \\
&\vdots \\
&= x - \left(\binom{3}{1} + \binom{3}{3}\right)\frac{x^3}{3!} + \left(\binom{5}{1} + \binom{5}{3} + \binom{5}{5}\right)\frac{x^5}{5!} - \left(\binom{7}{1} + \binom{7}{3} + \binom{7}{5} + \binom{7}{7}\right)\frac{x^7}{7!} \\
&\quad \left(\binom{9}{1} + \binom{9}{3} + \binom{9}{5} + \binom{9}{7} + \binom{9}{9}\right)\frac{x^9}{9!} \\
&\quad - \left(\binom{11}{1} + \binom{11}{3} + \binom{11}{5} + \binom{11}{7} + \binom{11}{9} + \binom{11}{11}\right)\frac{x^{11}}{11!} + \cdots \\
&= x - \frac{4x^3}{3!} + \frac{16x^5}{5!} - \frac{64x^7}{7!} + \frac{256x^9}{9!} - \frac{1024x^{11}}{11!} + \cdots
\end{aligned}$$

One can now easily verify that as far as we have calculated, the series for $\sin(2x)$ and that for $2\cos x \sin x$ agree.