§8.09

\# 5. $f(x)=\sin x$, $a=\frac{\pi}{4}$. $f(x)=\sin x$, $f^{(1)}(x)=\cos x$, $f^{(2)}(x)=-\sin x$, $f^{(3)}(x)=-\cos x$ $f^{(4)}(x)=\sin x$. Hence $f(\frac{\pi}{4})=\frac{\sqrt{2}}{2}$, $f^{(1)}(\frac{\pi}{4})=\frac{\sqrt{2}}{2}$, $f^{(2)}(\frac{\pi}{4})=-\frac{\sqrt{2}}{2}$, $f^{(3)}(\frac{\pi}{4})=-\frac{\sqrt{2}}{2}$ $f^{(4)}(\frac{\pi}{4})=\frac{\sqrt{2}}{2}$.

Hence

$P_0(x)=\frac{\sqrt{2}}{2}$; $P_1(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})$; $P_2(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})-\frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2$;
$P_3(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})-\frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2-\frac{\sqrt{2}}{12}(x-\frac{\pi}{4})^3$.

\# 7. $f(x)=\sqrt{x}$, $a=4$. $f(x)=x^{\frac{1}{2}}$; $f^{(1)}(x)=\frac{1}{2}x^{-\frac{1}{2}}$; $f^{(2)}(x)=-\frac{1}{4}x^{-\frac{3}{2}}$; $f^{(3)}(x)=\frac{3}{8}x^{-\frac{5}{2}}$; $f^{(4)}(x)=-\frac{15}{16}x^{-\frac{7}{2}}$.

$f(4)=2$; $f^{(1)}(4)=\frac{1}{4}$; $f^{(2)}(4)=-\frac{1}{32}$; $f^{(3)}(4)=\frac{3}{256}$; $f^{(4)}(4)=-\frac{15}{2048}$.

Hence

$P_0(x)=2$; $P_1(x)=2+\frac{1}{4}(x-4)$; $P_2(x)=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2$;
$P_3(x)=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3$.

\# 11. $\frac{1}{1+x}$. The series $\sum_{n=0}^{\infty}(-1)^n x^n=\frac{1}{1+x}$ so this series is the Maclaurin series. With more work, we can evaluate the Maclaurin series from the definition. First check that $f(x)=(1+x)^{-1}$ and $f^{(n)}(x)=(-1)(-2)\cdots(-n)(1+x)^{-n-1}=(-1)^n n!(1+x)^{-n-1}$ so again the Maclaurin series is $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{(-1)^n n!}{n!}x^n=\sum_{n=0}^{\infty}(-1)^n x^n$. Notice that this calculation of the Maclaurin series does less than the first argument since the first argument also shows that the Maclaurin series converges to the function on the interval $(-1,1)$.

\# 17. Given that e^x has the power series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$, it follows that $e^{-x}=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$ so $\cosh x=\frac{e^x+e^{-x}}{2}$ can be found by adding the two power series to get $\cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$. Hence this is the Maclaurin series and the series represents the function everywhere since the same is true for e^x. We can also compute the Maclaurin series via the definition as follows. $f(x)=\cosh x$; $f^{(1)}(x)=\sinh x$; $f^{(2)}(x)=\cosh x$. Hence $f^{(2n+1)}(x)=\sinh x$ and $f^{(2n)}(x)=\cosh x$. It follows that $f^{(2n)}(0)=1$ and $f^{(2n+1)}(0)=0$, so the Maclaurin series for $\cosh x$ is $\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

\# 33. The linearization at $x=a$ is always $f(a)+f'(a)(x-a)$ and the quadratic approximation is always $f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$. For $f(x)=\ln(\cos x)$ and $a=0$ we can compute as follows: $f'(x)=\frac{\frac{d\cos x}{dx}}{\cos x}=\frac{-\sin x}{\cos x}=-\tan x$; $f''(x)=\frac{d-\tan x}{dx}=-\sec^2 x$. Hence at 0, $f(0)=0$, $f'(0)=0$ and $f''(0)=-1$. Hence the linearization is 0 and the quadratic approximation is $-x^2/2$.