\# 1. Write the first four terms of the binomial series for $(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n$. The first four terms are $1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3$, or $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$.

\# 5. Write the first four terms of the binomial series for $(1+\frac{x}{2})^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{x}{2}\right)^n$. The first four terms are $1 - 2\,\frac{x}{2} + \frac{-2(-2-1)}{2!}\left(\frac{x}{2}\right)^2 + \frac{-2(-2-1)(-2-2)}{3!}\left(\frac{x}{2}\right)^3$, or $1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3$.

\# 21. Solve $y' - xy = 0$ with $y(0) = 1$. We are looking for a power series solution, $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$ and $xy = \sum_{n=0}^{\infty} a_n x^{n+1}$ so $y' - xy = a_1 + \sum_{n=1}^{\infty} \big((n+1)a_{n+1} - a_{n-1}\big)x^n$. The only way a power series can vanish is if all its coefficients vanish, so $a_1 = 0$ and the recursion relation $(n+1)a_{n+1} - a_{n-1} = 0$ holds. The initial value condition $y(0) = 1$ shows $a_0 = 1$. The recursion relation can be rewritten $(k+2)a_{k+2} = a_k$, or $a_{k+2} = \frac{a_k}{k+2}$. Since $a_1 = 0$, $a_{2k+1} = 0$ for all integers $k \geq 0$. Since $a_0 = 1$, $a_{2k} = \frac{1}{2^k k!}$ for all integers $k \geq 0$. Hence $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$ is the solution. One can see easily that $e^{\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$ as well. $y = e^{\frac{x^2}{2}}$ is the answer you will get if you apply your technique for solving first order linear equations to this problem.

\# 23. Solve$(1-x)y' - y = 0$ with $y(0) = 2$. Same yoga as the last problem. We are looking for a power series solution, $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$. Then $(1-x)y' = y' - xy' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} = a_1 + \sum_{n=1}^{\infty} \big((n+1)a_{n+1} - n a_n\big)x^n$ and $(1-x)y' - y = a_1 - a_0 + \sum_{n=1}^{\infty} \big((n+1)a_{n+1} - n a_n - a_n\big)x^n = a_1 - a_0 + \sum_{n=1}^{\infty} (n+1)(a_{n+1} - a_n)x^n$. In order for this power series to vanish, $a_1 - a_0 = 0$ and $a_{n+1} - a_n = 0$ for all $n \geq 1$. Hence $a_0 = a_1 = a_2 = \cdots = a_n = \cdots$. From the initial value condition $y(0) = 2$ we see $a_0 = 2$, so $y = \sum_{n=0}^{\infty} 2x^n$ is our solution. Note that this series is a geometric series so $y = \frac{2}{1-x}$ is another expression for the solution.

43. Find a polynomial that will approximate the function $F(x) = \int_0^x \sin t^2 \, dt$ on the interval $[0,1]$ with an error of magnitude less than 10^{-3}. First write down an exact power series solution: $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ so $\sin t^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (t^2)^{2n+1}}{(2n+1)!} =$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (t)^{4n+2}}{(2n+1)!} \text{ and } F(x) = \int_0^x \sin t^2 dt = \sum_{n=0}^{\infty} \frac{(-1)^n \left.\frac{(t)^{4n+3}}{4n+3}\right|_0^x}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)\cdot(2n+1)!}.$$

On the interval $[0,1]$ the series is alternating because $x^{4n+3} \geq 0$ and one can see that for a fixed x in this interval $\frac{x^{4n+3}}{(4n+3)\cdot(2n+1)!}$ decreases to 0. Hence the estimate associated to the Alternating Series Test applies.

This says that the polynomial of degree m, $\sum_{n=0}^{m} \frac{(-1)^n x^{4n+3}}{(4n+3)\cdot(2n+1)!}$ has the property that $\left| F(x) - \sum_{n=0}^{m} \frac{(-1)^n x^{4n+3}}{(4n+3)\cdot(2n+1)!} \right| \leq \frac{x^{4m+7}}{(4m+7)\cdot(2m+3)!}$ for every x in the interval $[0,1]$. Since x^{4m+7} increases as x does, $\left| F(x) - \sum_{n=0}^{m} \frac{(-1)^n x^{4n+3}}{(4n+3)\cdot(2n+1)!} \right| \leq \frac{1}{(4m+7)\cdot(2m+3)!}$ and we may choose any m so that $\frac{1}{(4m+7)\cdot(2m+3)!} < 10^{-3}$. When $m = 1$, $(4m+7)\cdot(2m+3)! = 11 \cdot 5! = 11 \cdot 120 = 1320 > 1000$. Hence the polynomial $\frac{x^3}{3} - \frac{x^7}{42}$ approximates $F(x)$ to within 10^{-3} on the interval $[0,1]$. The answer in the back of the book is not completely correct. If one is approximating $F(x)$ one would like to see a polynomial in x, not t. The book is not wrong to include an extra term. Once the polynomial of degree m works, so does the polynomial of degree $m+1$.

53. Find $\lim_{x\to\infty} x^2(e^{\frac{-1}{x^2}} - 1)$. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $e^{\frac{-1}{x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\ x^{2n}}$ so $e^{\frac{-1}{x^2}} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\ x^{2n}}$ and hence $x^2\big(e^{\frac{-1}{x^2}} - 1\big) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\ x^{2n-2}} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(n+1)!\ x^{2n}}$. As $x \to \infty$, $\frac{1}{x^2}$ goes to 0 so only the degree 0 term is left in the limit and that term is -1.

Hence $\lim_{x\to\infty} x^2(e^{\frac{-1}{x^2}} - 1) = -1$.