1.(5pt) *Which formulae below is* $\frac{d(\ln u)}{dx}$ *?*

(a)$\frac{\ln u}{u}$ (b)$\frac{1}{x}$ (c)$\frac{1}{\frac{du}{dx}}$ (d)$\frac{1}{u}\frac{du}{dx}$ (e)$\frac{1}{x}\frac{du}{dx}$

This is memorization. Either you know that (d) is the answer, or else you remember $\frac{d(\ln x)}{dx} = \frac{1}{x}$ and then use the Chain Rule: $\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$.

2.(5pt) *Which formulae below is* $\frac{d(\arcsin v)}{dt}$ *?*

(a)$\frac{v}{\sqrt{v^2-1}}$ (b)$\frac{1}{\sqrt{1-t^2}}$ (c)$\frac{\frac{dv}{dt}}{v\sqrt{v^2-1}}$ (d)$\frac{v\frac{dv}{dt}}{\sqrt{1-v^2}}$ (e)$\frac{\frac{dv}{dt}}{\sqrt{1-v^2}}$

Again this is memorization. Either you know that (e) is the answer, or else you remember $\frac{d(\arcsin x)}{dx} = \frac{1}{\sqrt{1-x^2}}$ and then use the Chain Rule: $\frac{d(\arcsin v)}{dx} = \frac{1}{\sqrt{1-v^2}}\frac{dv}{dx}$.

3.(20pt) *An unknown element is found to be radioactive. By chemical analysis, it is found that 15% of a given sample is lost in a 24 hour period. What is the half-life of this element? (An answer such as* $t = \frac{\ln 97}{\ln 78.64}$ *is fine.)*

Since we have radioactive decay, $y = y_0e^{-kt}$ where t is the time in days from now. The first condition says $y(1) = 0.85y(0)$, so $e^{-k\cdot 1} = 0.85e^{k\cdot 0}$, or $e^{-k} = 0.85$. The half-life h is given by the equation $y(h) = 0.5y(0)$, so $e^{-k\cdot h} = 0.5$. Hence $-k\cdot h = \ln(0.5)$ so $h = \frac{\ln(0.5)}{-k}$. But $e^{-k} = 0.85$ so $-k = \ln(0.85)$ and hence $h = \frac{\ln(0.5)}{\ln(0.85)}$ days.

If you prefer, you can measure t is hours. Then $y(24) = y_0e^{-k\cdot 24} = 0.85y_0e^{k\cdot 0}$ so $e^{-k\cdot 24} = 0.85$ and hence $-24k = \ln(0.85)$ or $-k = \frac{\ln(0.5)}{24}$. The half-life equation is still $y(h) = 0.5y(0)$ so $h = \frac{\ln(0.5)}{-k}$ and hence $h = \frac{\ln(0.5)}{\frac{\ln(0.5)}{24}} = 24\,\frac{\ln(0.5)}{\ln(0.85)}$ hours.

4.(10pt) *Evaluate* $\lim_{y\to 0}\dfrac{y^2}{\cos(3y)-1} =$

This is a L'Hôpital's Rule problem. Since y^2 is 0 and $\cos(3y) = 1$ when we plug in $y = 0$, this limit has the form $\frac{0}{0}$. Hence $\lim_{y\to 0}\dfrac{y^2}{\cos(3y)-1} = \lim_{y\to 0}\dfrac{2y}{(-3)\sin(3y)}$. This second limit has the form $\frac{0}{0}$ as well, since $2y$ and $\sin(3y)$ are both 0 when $y = 0$ is plugged in. By L'Hôpital's Rule again, $\lim_{y\to 0}\dfrac{2y}{(-3)\sin(3y)} = \lim_{y\to 0}\dfrac{2}{(-3)(3)\cos(3y)}$ and this last limit can be evaluated by plugging in $y = 0$. Since $\cos(3\cdot 0) = 1$, we get $\frac{2}{(-3)(3)} = -\frac{2}{9}$.

5.(20pt) *Evaluate*

a)$\displaystyle\int\frac{dv}{16+9v^2} =$

One can proceed in several similar ways: e.g. factor out 16 to get $\displaystyle\int\frac{dv}{16+9v^2} = \frac{1}{16}\int\frac{dv}{1+\frac{9}{16}v^2}$. Now substitute $w = \frac{3}{4}v$ ($dw = \frac{3}{4}dv$ or $\frac{4}{3}dw = dv$) so $\displaystyle\frac{1}{16}\int\frac{dv}{1+\frac{9}{16}v^2} = \frac{1}{16}\int\frac{4}{3}\frac{dw}{1+w^2} = \frac{1}{12}\int\frac{dw}{1+w^2} = \frac{1}{12}\arctan w + C = \frac{1}{12}\arctan\Big(\frac{3}{4}v\Big) + C$. Another way is to use the formula $\displaystyle\int\frac{dw}{a^2+w^2} = \frac{1}{a}\arctan(\frac{w}{a}) + C$. This formula shows $\displaystyle\int\frac{d(3v)}{4^2+(3v)^2} = \frac{1}{4}\arctan\Big(\frac{3v}{4}\Big) + C$ and we also have $\displaystyle\int\frac{dv}{16+9v^2} = \frac{1}{3}\int\frac{d(3v)}{4^2+(3v)^2}$ so we get the same answer.

b)$\displaystyle\int\frac{vdv}{16+9v^2} =$

This is also a substitution: let $w = 16+9v^2$ ($dw = 18v\,dv$) so $dv = \frac{1}{18}\,dw$. Hence $\displaystyle\int\frac{vdv}{16+9v^2} = \int\frac{\frac{1}{18}dw}{w} = \frac{1}{18}\int\frac{dw}{w} = \frac{1}{18}\ln|w| + C = \frac{1}{18}\ln|16+9v^2| + C$.

6.(15pt) *Find the unique function satisfying the two conditions* $\frac{dy}{dt} = 4y$ *and* $y(1) = 3$.

The first step is to solve the differential equation. This is the growth/decay equation again. You can remember the formula for the answer, $y = y_0 e^{4t}$, or else derive the formula as follows. From $\frac{dy}{dt} = 4y$, we have $\frac{dy}{y} = 4dt$ so $\int \frac{dy}{y} = \int 4dt$ and hence $\ln|y| = 4t + C$. Solving this equation, $|y| = e^{4t+C} = e^C \cdot e^{4t}$ and hence $y = y_0 e^{4t}$.

Next turn to the initial value part of the problem. From $y(1) = 3$, we see $y(1) = y_0 e^{4\cdot 1}$ so $y_0 e^4 = 3$ and $y_0 = \frac{3}{e^4}$ so $y = \frac{3}{e^4} e^{4t}$ or $3e^{4t-4}$.

7.(10pt) *Evaluate* $\displaystyle\int \frac{du}{\sqrt{4u - u^2 - 3}} =$

First complete the square: $4u - u^2 - 3 = 1 - (u-2)^2$ so $\displaystyle\int \frac{du}{\sqrt{4u - u^2 - 3}} = \int \frac{du}{\sqrt{1-(u-2)^2}}$. Now substitute $w = u - 2$ ($dw = du$) so $\displaystyle\int \frac{du}{\sqrt{1-(u-2)^2}} = \int \frac{dw}{\sqrt{1-w^2}} = \arcsin w + C = \arcsin(u-2) + C$.

8.(15pt) $\displaystyle\int_{\ln 2}^{\ln 5} \frac{dy}{\sqrt{e^{2y}-1}} =$

Substitute $u = e^y$ so $e^{2y} = u^2$. Now $du = e^y dy$ or $\frac{du}{u} = dy$, so $\displaystyle\int_{\ln 2}^{\ln 5} \frac{dy}{\sqrt{e^{2y}-1}} = \int_2^5 \frac{\frac{du}{u}}{\sqrt{u^2-1}} = \int_2^5 \frac{du}{u\sqrt{u^2-1}} = \text{arcsec } |u| \Big|_2^5 = \text{arcsec } (5) - \text{arcsec } (2)$. Now evaluate $\text{arcsec } (2) = \pi/3$.

Alternatively, you can factor e^y out of the square root: $\int \frac{dy}{\sqrt{e^{2y}-1}} = \int \frac{dy}{e^y\sqrt{1-e^{-2y}}} = \int \frac{e^{-y}dy}{\sqrt{1-e^{-2y}}}$. Now make the substitution $u = e^{-y}$ (so $du = -e^{-y}dy$); the integral is then $\int_{\frac{1}{2}}^{\frac{1}{5}} \frac{-du}{\sqrt{1-u^2}} = -\arcsin(u)\Big|_{\frac{1}{2}}^{\frac{1}{5}} = \arcsin(1/2) - \arcsin(1/5) = \pi/6 - \arcsin(1/5)$.

The original problem was $\displaystyle\int_0^{\ln 5} \frac{dy}{\sqrt{e^{2y}-1}}$ but as some of you noticed, the integrand blows up at 0. Hence this integral is what is called an "improper integral" and by definition $\displaystyle\int_0^{\ln 5} \frac{dy}{\sqrt{e^{2y}-1}} = \lim_{t\to 0^+} \int_t^{\ln 5} \frac{dy}{\sqrt{e^{2y}-1}}$ As above $\displaystyle\lim_{t\to 0^+} \int_t^{\ln 5} \frac{dy}{\sqrt{e^{2y}-1}} = \lim_{t\to 1^+} \int_t^5 \frac{du}{u\sqrt{u^2-1}} = \text{arcsec } (5) - \lim_{t\to 1^+} \text{arcsec } |t| = \text{arcsec } (5) - \text{arcsec } (1) = \text{arcsec } (5)$.