Math 126
Fall 1997
Test 2
Oct. 28, 1997

1. (5pt) $\int_0^1 2xe^{x^2}\,dx = ?$ | d |

(a) 0 (b) 1 (c) e (d) $e-1$ (e) e^2

Substitute $u = x^2$ so $du = 2x\,dx$. Then $\int_0^1 2xe^{x^2}\,dx = \int_0^1 e^u du = e^u \Big|_0^1 = e^1 - e^0 = e - 1$.

Hence (d) is the correct answer.

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2. (5pt) $\int_0^4 \frac{dx}{(4-x)^{\frac{3}{2}}} = ?$ | e |

(a) 1 (b) 2 (c) 3 (d) 4 (e) The integral diverges.

The integral is improper because the function we are integrating is undefined at 4. First do the indefinite integral. Substitute $u = 4 - x$, $du = -dx$. $\int \frac{dx}{(4-x)^{\frac{3}{2}}} = -\int \frac{du}{u^{\frac{3}{2}}} = -\int u^{-\frac{3}{2}}\,du = -\frac{u^{1-\frac{3}{2}}}{1-\frac{3}{2}} + C = \frac{2}{u^{\frac{1}{2}}} + C = \frac{2}{(4-x)^{\frac{1}{2}}} + C$. Now return to our problem. $\int_0^4 \frac{dx}{(4-x)^{\frac{3}{2}}} = \lim_{t\to 4^-} \int_0^t \frac{dx}{(4-x)^{\frac{3}{2}}} = \lim_{t\to 4^-}\left(\frac{2}{(4-t)^{\frac{1}{2}}} - \frac{2}{4^{\frac{1}{2}}}\right) = \infty - 1$, so the integral diverges and the answer is (e).

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3. (15pt) Evaluate $\int_0^1 \frac{dx}{\sqrt{2x-x^2}}$. First note that this integral is improper because the function we are integrating is undefined at 0. First we do the indefinite integral. There are several ways to do this. The first way starts by completing the square: $2x-x^2 = 1-(x-1)^2$. Now we can do a trig. substitution, $x-1 = \sin\theta$, $dx = \cos\theta\,d\theta$ and $2x-x^2 = 1-(x-1)^2 = 1-\sin^2\theta = \cos^2\theta$ and $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{\cos\theta\,d\theta}{\sqrt{\cos^2\theta}} = \int d\theta = \theta + C = \arcsin(x-1) + C$. A second way to do the integral is to use the substitution $u = \sqrt{\frac{x}{2}}$, $du = \frac{dx}{2\sqrt{2}\sqrt{x}}$ or $2\sqrt{2}\,du = \frac{dx}{\sqrt{x}}$. So $\int \frac{dx}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{1-\frac{x}{2}}}\frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{2}}\int \frac{1}{1-u^2}\,2\sqrt{2}\,du = 2\int \frac{du}{\sqrt{1-u^2}} = 2\arcsin u + C = 2\arcsin\left(\sqrt{\frac{x}{2}}\right) + C$. Note that these two answers are

the same - you have to show $\arcsin(x-1) = 2\arcsin\left(\sqrt{\frac{x}{2}}\right) - \frac{\pi}{2}$. Some students really did use this second method. Yet a third method was used. Substitute $x = 2\sin^2\theta$, $dx = 4\sin\theta\ \cos\theta\ d\theta$, $2x - x^2 = 4\sin^2\theta - 4\sin^4\theta = 4\sin^2\theta(1-\sin^2\theta) = 4\sin^2\theta\cos^2\theta$. Hence $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{4\sin\theta\ \cos\theta\ d\theta}{\sqrt{4\sin^2\theta\cos^2\theta}} = 2\int d\theta = 2\theta + C = 2\arcsin\left(\sqrt{\frac{x}{2}}\right) + C$.

Now turn to the improper integral. $\displaystyle\int_0^1 \frac{dx}{\sqrt{2x-x^2}} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{2x-x^2}} = \lim_{t\to 0^+}\arcsin(x-1)\Big|_t^1 = \lim_{t\to 0^+}\Big(\arcsin(0) - \arcsin(t-1)\Big) = \arcsin(0) - \arcsin(-1) = 0 - (-\frac{\pi}{2}) = \frac{\pi}{2}$. Hence the integral converges and has value $\frac{\pi}{2}$.

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4. (15pt) Evaluate $\displaystyle\int_0^1 (x+1)\ln(x+1)\,dx$.

There are several methods here too. The first method is to begin with an integration by parts: $u = \ln(x+1)$ $dw = (x+1)dx$. Hence $du = \dfrac{dx}{x+1}$ and $w = \frac{x^2}{2} + x$. Then $\displaystyle\int_0^1 (x+1)\ln(x+1)\,dx = \left(\frac{x^2}{2}+x\right)\ln(x+1)\Big|_0^1 - \int_0^1\left(\frac{x^2}{2}+x\right)\frac{dx}{x+1} = \frac{3}{2}\ln 2 - \frac{1}{2}\int_0^1 \frac{x^2+2x}{x+1}\,dx = \frac{3}{2}\ln 2 - \frac{1}{2}\int_0^1\left(x+1-\frac{1}{x+1}\right)dx = \frac{3}{2}\ln 2 - \frac{1}{2}\left(\frac{x^2}{2}+x-\ln(x+1)\right)\Big|_0^1 = \frac{3}{2}\ln 2 - \frac{1}{2}\left(\frac{1}{2}+1-\ln 2\right) = \frac{3}{2}\ln 2 + \frac{1}{2}\ln 2 - \frac{3}{4} = 2\ln 2 - \frac{3}{4}$.

A second method starts the same way $u = \ln(x+1)$ $dw = (x+1)dx$. Hence $du = \dfrac{dx}{x+1}$ but use $w = \dfrac{(x+1)^2}{2}$. Then $\displaystyle\int_0^1 (x+1)\ln(x+1)dx = \frac{(x+1)^2}{2}\ln(x+1)\Big|_0^1 - \int_0^1 \frac{(x+1)^2}{2}\frac{dx}{x+1} = \frac{4}{2}\ln 2 - 0 - \frac{1}{2}\int_0^1 (x+1)\,dx = 2\ln 2 - \frac{1}{2}(\frac{x^2}{2}+x)\Big|_0^1 = 2\ln 2 - \frac{1}{2}\cdot\frac{3}{2} = 2\ln 2 - \frac{3}{4}$.

A third method begins with the substitution $u = x+1$, $du = dx$ so $\displaystyle\int_0^1 (x+1)\ln(x+1)\,dx = \int_1^2 u\ln u\,du$ and attack $\displaystyle\int u\ln u\,du$ by parts: $z = \ln u$, $dw = u\,du$ so $dz = \frac{du}{u}$ $w = \frac{1}{2}u^2$ and $\displaystyle\int_1^2 u\ln u\,du = \frac{u^2}{2}\ln u\Big|_1^2 - \int_1^2 \frac{u^2}{2}\frac{du}{u} = 2\ln 2 - \frac{1}{2}\int_1^2 u\,du = 2\ln 2 - \frac{1}{2}(\frac{u^2}{2}\Big|_1^2) = 2\ln 2 - \frac{1}{2}\cdot(2-\frac{1}{2}) = 2\ln 2 - \frac{1}{2}\cdot\frac{3}{2} = 2\ln 2 - \frac{3}{4}$.

A fourth method starts with the formula $\displaystyle\int \ln z\,dz = z\ln z - z + C$. Now do the substitution

$u = x+1$, $du = dx$ again. Do $\int u \ln u \, du$ by parts again, but now use $dw = \ln u \, du$, $y = u$, so $w = u \ln u - u$ and $dy = du$ so $\int u \ln u \, du = u(u \ln u - u) - \int u \ln u \, du + \int u \, du$. Now solve for $\int u \ln u du$: $2 \int u \ln u du = u^2 \ln u - u^2 + \frac{u^2}{2} + C$ so $\int u \ln u du = \frac{u^2}{2} \ln u - \frac{u^2}{4} + C_1$. Now evaluate at 2 and 1.

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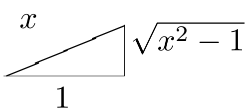
5. (15pt) Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}$.

Trig. substitute; $x = \sec\theta$ so $dx = \sec\theta \, \tan\theta \, d\theta$ and $x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$. Hence $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta \, \tan\theta \, d\theta}{\sec^2\theta \tan\theta} = \int \cos\theta \, d\theta = \sin\theta + C$. Solving the triangle



shows $\sin\theta = \frac{\sqrt{x^2-1}}{x}$ so $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x} + C$.

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6. (15pt) Evaluate $\displaystyle\int \frac{2dx}{(x-1)^2(x^2+1)}$.

Do partial fractions: $\frac{2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$ or $2 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$. Plug in $x = 1$, so $2 = B(2)$ or $B = 1$. Hence $2 - (x^2+1) = A(x-1)(x^2+1) + (Cx+D)(x-1)^2$ so $1 - x^2 = A(x-1)(x^2+1) + C(x-1)^2$ and $-(1+x) = A(x^2+1) + (Cx+D)(x-1)$. Plug in $x = 1$ so $-2 = A(2)$ or $A = -1$. Hence $-(1+x) + (x^2+1) = (Cx+D)(x-1)$ or $x^2 - x = (Cx+D)(x-1)$ or $x = Cx + D$ so $C = 1$, $D = 0$. Hence $\displaystyle\int \frac{2dx}{(x-1)^2(x^2+1)} = -\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} + \int \frac{x\,dx}{x^2+1} = -\ln|x-1| - \frac{1}{x-1} + \frac{1}{2}\ln(x^2+1) + C$.

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7. (15pt) Solve the initial value problem: $(1+x)y' + y = 1$; $y(0) = 0$.

This equation is both separable and linear. As a linear problem, the standard form is $y' + \frac{1}{x+1}y = \frac{1}{x+1}$. The integrating factor is found by integrating $\frac{1}{x+1}$ to get $\ln|x+1| + C$ so we may take $e^{\ln|x+1|}$ as the integrating factor. Even better, we may use $|x+1|$ or even $v = x+1$. Then $(vy)' = v \cdot \frac{1}{x+1}$ so $vy = \int dx = x + C$ or $y = \frac{x+C}{x+1}$. Since $y(0) = 0$ we also have $\frac{0+C}{0+1} = 0$ so $C = 0$ and $y = \frac{x}{x+1}$.

As a separable problem, $(x+1)\frac{d(y)}{dx} = 1 - y$ so $\frac{dy}{1-y} = \frac{dx}{x+1}$. Integrating $\displaystyle\int \frac{dy}{1-y} = \int \frac{dx}{x+1}$ and $-\ln|1-y| = \ln|x+1| + C$. Clearing the log's, $\left|1-y\right| = e^C\left|\frac{1}{x+1}\right|$ or $1 - y = A \cdot \frac{1}{x+1}$ for some constant A. Hence $y = 1 - \frac{A}{x+1}$. Since $y(0) = 0$, $0 = 1 - \frac{A}{0+1}$ or $A = 1$. Hence $y = 1 - \frac{1}{x+1}$ which is the same function as before.

8a. (9pt) Is the integral $\int_0^\infty \frac{dx}{e^x}$ convergent or divergent? If it is convergent, evaluate it.

8b. (6pt) Is the integral $\int_0^\infty \frac{dx}{1+e^x}$ convergent or divergent? Do not evaluate the integral.

First do the indefinite integral $\int \frac{dx}{e^x} = \int e^{-x}dx = -e^{-x} + C$.

Hence $\int_0^\infty \frac{dx}{e^x} = \lim_{t\to\infty} \int_0^t \frac{dx}{e^x} = \lim_{t\to\infty} \left(-e^{-x}\right)\Big|_0^t = \lim_{t\to\infty} -e^{-t} - (-e^{-0}) = 1 - \lim_{t\to\infty} e^{-t} = 1 - 0$ so the improper integral converges and the value is 1.

For part (b) use either comparison theorem. To use either, note first that $\frac{1}{1+e^x} > 0$. To use the first comparison theorem note that $1 + e^x > e^x$ so $\frac{1}{1+e^x} < \frac{1}{e^x}$ and since the integral in part (a) converges, so does $\int_0^\infty \frac{dx}{1+e^x}$. One can even argue that its value is positive and less than 1.

To use the second comparison theorem compute $\lim_{x\to\infty} \frac{\frac{1}{e^x}}{\frac{1}{1+e^x}} = \lim_{x\to\infty} \frac{1+e^x}{e^x} = \lim_{x\to\infty} 1 + e^{-x} = 1 + 0 = 1$ so both $\frac{1}{1+e^x}$ and $\frac{1}{e^x}$ grow at the same rate. Since the integral in part (a) converges, so does $\int_0^\infty \frac{dx}{1+e^x}$.

One can actually do the second integral: use the substitution $u = e^x$, $du = e^x dx$ or $\frac{du}{u} = dx$ so $\int \frac{dx}{1+e^x} = \int \frac{du}{u(1+u)}$. Now use partial fractions: $\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$ or $1 = A(u+1) + Bu$. Plug in $u = 0$ so $A = 1$ and plug in $u = -1$ so $B = -1$ and $\int \frac{du}{u(1+u)} = \ln|u| - \ln|u+1| + C = \ln\left|\frac{u}{(u+1)}\right| + C = \ln\left|\frac{e^x}{(e^x+1)}\right| + C$. Hence $\int_0^\infty \frac{dx}{1+e^x} = \lim_{t\to\infty} \int_0^t \frac{dx}{1+e^x} = \lim_{t\to\infty} \ln\left|\frac{e^t}{(e^t+1)}\right| - \ln\left|\frac{e^0}{(e^0+1)}\right| = \ln\left|\lim_{t\to\infty} \frac{e^t}{(e^t+1)}\right| - \ln\frac{1}{2} = \ln\left|\lim_{t\to\infty} \frac{e^t}{(e^t+1)}\right| + \ln 2 = \ln 1 + \ln 2 = \ln 2.$