Math 126
Fall 1997
Test 3
Dec. 2, 1997

1. (5pt) The Maclaurin series for $\sin(2x^2)$ starts off as: **a**

(a) $2x^2 - \frac{4x^6}{3} + \frac{4x^{10}}{15} - \ldots$ (b) $1 - \frac{2x^4}{2!} + \frac{2x^8}{4!} - \ldots$ (c) $2x^2 - \frac{2x^6}{3!} + \frac{2x^{10}}{5!} - \ldots$

(d) $1 - 2x^4 + \frac{2x^8}{3} - \ldots$ (e) none of the above

Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, then $\sin(2x^2) = 2x^2 - \frac{(2x^2)^3}{3!} + \frac{(2x^2)^5}{5!} - \cdots = 2x^2 - \frac{8x^3}{6} + \frac{32x^{10}}{120} - \cdots = 2x^2 - \frac{4x^3}{3} + \frac{4x^{10}}{15} - \cdots$.

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2. (5pt) $\sum_{n=0}^{\infty} \frac{3}{(n+2)(n+3)} = ?$ **d**

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{3}{2}$ (e) diverges

This is a telescoping series: a partial fraction decomposition gives $\frac{3}{(n+2)(n+3)} = \frac{3}{n+2} - \frac{3}{n+3}$, so the sum is $(\frac{3}{2} - \frac{3}{3}) + (\frac{3}{3} - \frac{3}{4}) + (\frac{3}{4} - \frac{3}{5}) + \cdots$. Everything cancels except for the first term, $\frac{3}{2}$.

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3. (10pt) For each of the following series, does it converge or diverge? If it converges, identify the sum. Give reasons for your answers.

(a) $\sum_{n=0}^{\infty} (-1)^{n+1}\pi^{-n}$.

We can rewrite the sum as $\sum(-1)(-1)^n\pi^{-n} = -\sum(\frac{-1}{\pi})^n$. This sum is a geometric series with ratio $-\frac{1}{\pi}$; the ratio has absolute value less than 1, so it converges, and the sum is $\frac{-1}{1-(-\frac{1}{\pi})} = \frac{-\pi}{\pi+1}$. (You can also use the alternating series test to see that it converges, or the ratio or nth root test to see that it converges absolutely.)

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

This is a p-series with $p = \frac{1}{2} < 1$, so it diverges. (You can also use the integral test to see this. You can also compare the series to $\frac{1}{n}$: $\frac{1}{\sqrt{n}} > \frac{1}{n}$, and $\sum \frac{1}{n}$ diverges.)

4. (20pt) Determine whether each of the following series converges absolutely, converges conditionally, or diverges. Give reasons for your answers.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3n-2}$

This series converges conditionally. It is an alternating series (at least when $n \geq 1$), the terms go to zero, and their absolute values are decreasing. So the series converges by the alternating series test. On the other hand, if one takes absolute values, the resulting series diverges: compare it to $\frac{1}{n}$ using the limit comparison test: $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{3n-2}}{\frac{1}{n}} = \frac{1}{3}$. Since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{1}{3n-2}$. (You can also use the integral test to see this. You can also compare the series to $\frac{1}{3n}$: $\frac{1}{3n-2} > \frac{1}{3n}$, and $\sum \frac{1}{3n}$ diverges.)

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{3n^3-2}$

This series converges absolutely. Compare it to $\sum \frac{1}{n^2}$: using the limit comparison test, you get $\displaystyle\lim_{n\to\infty} \frac{\frac{n}{3n^3-2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3}{3n^3-2} = \frac{1}{3}$. Therefore $\sum \frac{n}{3n-2}$ behaves the same way that $\sum \frac{1}{n^2}$ does. The latter is a p-series with $p = 2$, and hence converges. Therefore the former converges, and therefore the alternating series $\sum \frac{(-1)^n n}{3n-2}$ converges absolutely. (You can also use the direct comparison test: $\frac{1}{n^2} > \frac{n}{3n^3-2}$ for all $n \geq 1$; since $\sum \frac{1}{n^2}$ converges and is larger, term-by-term, than $\sum \frac{n}{3n^3-2}$, then this series converges, too.)

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^n$

This series converges absolutely. Use the nth root test: $\displaystyle\lim_{n\to\infty} \left[\left(\frac{3}{n}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{3}{n} = 0$. Since the limit is less than 1, then the series converges. Since all the terms are positive, it converges absolutely. (You can also use the ratio test, but it's more complicated. Or you can use the direct comparison test: when n is at least 10, then $\left(\frac{3}{n}\right)^n \leq \left(\frac{3}{10}\right)^n$. The geometric series $\sum \left(\frac{3}{10}\right)^n$ converges, and hence so does $\sum \left(\frac{3}{n}\right)^n$.)

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{\sqrt{n!}}$

This series converges absolutely. Use the ratio test on the absolute values of the terms: $\displaystyle\lim_{n\to\infty} \frac{(n+1)^2}{\sqrt{(n+1)!}} \frac{\sqrt{n!}}{n^2} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{1}{\sqrt{n+1}} = 0$. Since this is less than 1, the series of positive terms converges, so the original series converges absolutely.

5. (15pt) What is the interval of convergence of $\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}$?

Use the ratio test: $\lim_{n\to\infty} \frac{|2x|^{n+1}}{(n+1)^2} \frac{n^2}{|2x|^n} = \lim_{n\to\infty} |2x| \left(\frac{n}{n+1}\right)^2 = |2x|$. The series converges when this is less than 1, which is when $|x| < \frac{1}{2}$, or equivalently, when $-\frac{1}{2} < x < \frac{1}{2}$. We deal with the endpoints separately: when we plug $x = \frac{1}{2}$ into the series, we get $\sum \frac{1}{n^2}$ which converges because it is a p-series with $p = 2$. When we plug in $x = -\frac{1}{2}$, we get $\sum \frac{(-1)^n}{n^2}$, which converges by the absolute convergence test or the alternating series test. So the interval of convergence is $[-\frac{1}{2}, \frac{1}{2}]$: the series converges for all x with $-\frac{1}{2} \le x \le \frac{1}{2}$. (You can also use the nth root test to find the initial interval.)

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6. (15pt) Find the Taylor series for $f(x) = x^2 + e^x$ about $x = 1$.
We start by computing the derivatives of the function $f(x)$. We have: $f'(x) = 2x + e^x$, $f''(x) = 2 + e^x$, and $f'''(x) = e^x = f^{(4)}(x) = f^{(5)}(x) = \cdots$. Now we plug in the center of the series: $f(1) = 1+e$, $f'(1) = 2+e$, $f''(1) = 2+e$, and $f^{(n)}(1) = e$ for all $n \ge 3$. So the answer is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = (1+e)+(2+e)(x-1)+\frac{2+e}{2!}(x-1)^2+\frac{e}{3!}(x-1)^3+\frac{e}{4!}(x-1)^4+\cdots.$$

Alternatively, you can write this as

$$\left(1+2(x-1)+\frac{2}{2!}(x-1)^2\right)+e\left(1+(x-1)+\frac{1}{2!}(x-1)^2+\frac{1}{3!}(x-1)^3+\cdots\right).$$

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7. (15pt) Express $\int_0^1 e^{-x^2}\,dx$ as an infinite series. How accurate an estimate do you get from the first four terms of this series?
First, since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$. Therefore $\int e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} + c$. When we evaluate at 0 and 1, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}$.
The first four terms of this are $1 - \frac{1}{3\cdot 1!} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!}$. Since this series satisfies the conditions of the alternating series test (the terms alternate, they decrease in absolute value, and their limit is zero), the absolute value of the next term measures the error: $\frac{1}{9\cdot 4!} = \frac{1}{9\cdot 24} = \frac{1}{216}$.

8. (15pt) Use the binomial theorem to evaluate $\lim_{x\to 0} \dfrac{(1+x^2)^{3/4} - (1+\frac{3}{4}x^2)}{x^4}$.

The binomial theorem says that $(1+y)^{3/4} = 1 + \frac{3}{4}y - \frac{3}{32}y^2 + \frac{5}{128}x^8 - \cdots$. Therefore, $(1+x^2)^{3/4} = 1 + \frac{3}{4}x^2 - \frac{3}{32}x^4 + \frac{5}{128}x^8 - \cdots$. So the limit is

$$\lim_{x\to 0} \frac{(1+\frac{3}{4}x^2 - \frac{3}{32}x^4 + \frac{5}{128}x^8 - \cdots) - (1+\frac{3}{4}x^2)}{x^4} = \lim_{x\to 0} \frac{-\frac{3}{32}x^4 + \frac{5}{128}x^8 - \cdots}{x^4} = -\frac{3}{32}.$$