1. (15pt) Find the area inside the 3–leafed rose, $r = \sin(3\theta)$.

Area$=\frac{1}{2}\displaystyle\int_{\alpha}^{\beta} r^2\, d\theta$. From the graph check that the rose is swept out once as θ runs from 0 to π: 0 to $\frac{\pi}{3}$ is the leaf in the first quadrant; $\frac{\pi}{3}$ to $\frac{2\pi}{3}$ is the leaf centered on the negative y–axis; and $\frac{2\pi}{3}$ to π is the leaf in the second quadrant.

Hence **Area** $= \frac{1}{2}\displaystyle\int_0^{\pi} \sin^2(3\theta)\, d\theta = \frac{1}{2}\int_0^{\pi} \frac{1-\cos(6\theta)}{2}\, d\theta = \frac{1}{4}\left(\theta\Big|_0^{\pi} - \frac{1}{6}\sin(6\theta)\Big|_0^{\pi}\right) = \frac{\pi}{4}$.

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2. (15pt) Find the arclength of the curve $r = \sec\theta$ with $0 \le \theta \le \frac{\pi}{4}$.

There are two solutions. The first is to note $r = \sec\theta$ is equivalent to $r\cos\theta = 1$, or $x = 1$, so our graph is a part of the vertical line $x = 1$. The y–coordinate is given by $y = r\sin\theta = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$. When $\theta = 0$, $y = 0$ and when $\theta = \frac{\pi}{4}$, $y = 1$. Hence the length is 1.

The overwhelming majority of you proceeded as follows. **Length**$=\displaystyle\int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2}\, d\theta$.

$\dfrac{dr}{d\theta} = \sec\theta\tan\theta$; $r^2 + (r')^2 = \sec^2\theta + \sec^2\theta\tan^2\theta = \sec^{\theta}(1+\tan^2\theta) = \sec^2\theta\sec^2\theta$. Hence

Length$=\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\sec^4\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \sec^2\theta\, d\theta = \tan\theta\Big|_0^{\frac{\pi}{4}} = 1 - 0 = 1.$

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3. (15pt) Find the surface area of the surface obtained by rotating the piece of $r^2 = 1+\cos(2\theta)$ in the first quadrant around the x–axis. (The graph is that of the entire curve.)

By studying the graph, see that we need to rotate the part of the curve $r = \sqrt{1+\cos(2\theta)}$ for θ between 0 and π. **Surface Area** $= 2\pi\displaystyle\int_{\alpha}^{\beta} r\sin\theta\sqrt{r^2 + (r')^2}\, d\theta$.

$\dfrac{dr}{d\theta} = \dfrac{-2\sin(2\theta)}{2\sqrt{1+\cos(2\theta)}} = \dfrac{-\sin(2\theta)}{\sqrt{1+\cos(2\theta)}}$, so

$r^2 + (r')^2 = 1 + \cos(2\theta) + \dfrac{\sin^2(2\theta)}{1+\cos(2\theta)} = \dfrac{1 + 2\cos(2\theta) + \cos^2(2\theta) + \sin^2(2\theta)}{1+\cos(2\theta)} =$

$\dfrac{2 + 2\cos(2\theta)}{1+\cos(2\theta)} = 2$. Hence **Surface Area** $= 2\pi \displaystyle\int_0^{\frac{\pi}{2}} \sqrt{1+\cos(2\theta)}\ (\sin\theta)\sqrt{2}\, d\theta =$

$2\pi \displaystyle\int_0^{\frac{\pi}{2}} \sqrt{2\cos^2\theta}\ (\sin\theta)\sqrt{2}\, d\theta = 4\pi \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta\, d\theta = 2\pi \sin^2\theta \Big|_0^{\frac{\pi}{2}} = 2\pi.$

A second approach proceeds as follows. Rewrite $r = \sqrt{1+\cos(2\theta)}$ as $r = \sqrt{2}\ \cos\theta$ using the half-angle formula. Then $r' = -\sqrt{2}\ \sin\theta$ so $r^2 + (r')^2 = 2$ and **Surface Area** $= 2\pi \displaystyle\int_0^{\frac{\pi}{2}} (\sqrt{2}\ \cos\theta)\sin\theta\ \sqrt{2}\, d\theta$ and finish as above.

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4. (5pt) Which function below is the inverse function to $f(x) = e^{2x}$?

(a) $e^{-x} + \ln|x|$ (b) $e^{-x/2}$ (c) $\sqrt{\ln x}$ (d) $\bullet \frac{1}{2}\ln x$ (e) $\ln(x/2)$

Since $f(\frac{1}{2}\ln x) = e^{2\cdot\frac{1}{2}\ln x} = e^{\ln x} = x$, $\frac{1}{2}\ln x$ is the inverse function.

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5. (5pt) Which substitution reduces the integral $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$ to the integral $\displaystyle\int du$?

(a) $\bullet x = 2\sin u$ (b) $x = \frac{1}{2}\tan u$ (c) $x = \sin u + \cos u$ (d) $u = 2\sin x$ (e) $x = \sqrt{2}\cos u$

If $x = 2\sin u$, $dx = 2\cos u\ du$ and $\sqrt{4-x^2} = \sqrt{4 - 4\sin^2 u} = 2\sqrt{cos^2 u} = 2\cos u$, so $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\cos u\, du}{2\cos u}$.

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6. (5pt) Use Integration by Parts to show $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{10} x\, dx$ equals one of the numbers below.

(a) $\frac{\pi}{2}$ (b) $(-10)\displaystyle\int_0^{\frac{\pi}{2}} \cos^9 x \sin x\, dx$ (c) $\bullet \frac{9}{10}\displaystyle\int_0^{\frac{\pi}{2}} \cos^8 x\, dx$ (d) 0 (e) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^8 x\, dx$

$u = \cos^9 x$ $dv = \cos x\ dx$ so $du = -9\cos^8 x \sin x\ dx$, $v = \sin x$ and $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{10} x\ dx =$ $\cos^9 x \sin x \Big|_0^{\frac{\pi}{2}} + 9\displaystyle\int_0^{\frac{\pi}{2}} \cos^8 x \sin^2 x\ dx = 9\int_0^{\frac{\pi}{2}} \cos^8 x(1 - \cos^2 x)dx = 9\int_0^{\frac{\pi}{2}} \cos^8 x\ dx -$ $9\displaystyle\int_0^{\frac{\pi}{2}} \cos^{10} x\ dx$. Now solve for $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{10} x\ dx$.

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7. (5pt) The value of $\displaystyle\lim_{t\to\infty} t \cdot \sin\left(\frac{1}{2t}\right) =$

(a) 4 (b) 2 (c) 1 (d) $\frac{1}{4}$ (e) $\bullet \frac{1}{2}$

By L'Hôpital's Rule, $\displaystyle\lim_{t\to\infty} t \cdot \sin\left(\frac{1}{2t}\right) = \lim_{t\to\infty} \frac{\sin\left(\frac{1}{2t}\right)}{\frac{1}{t}} = \lim_{t\to\infty} \frac{\cos\left(\frac{1}{2t}\right)\frac{1}{2}\frac{-1}{t^2}}{\frac{-1}{t^2}} =$

$\displaystyle\frac{1}{2}\lim_{t\to\infty} \cos\left(\frac{1}{2t}\right) = \frac{1}{2}.$

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8. (5pt) Which function below is the solution to the initial value problem $y' = 3y$, $y(1) = 1$?
(a) $y(x) = \frac{e^3}{e^{3x}}$ (b) $\bullet y(t) = \frac{e^{3t}}{e^3}$ (c) $y(s) = e^{3s} - e^3 + 1$ (d) $y(x) = x^3$ (e) $y(t) = e^{3t}$
This is a growth/decay differential equation so $y = Ce^{3t}$. Since $y(1) = 1$, $C = \frac{1}{e^3}$.

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9. (5pt) Indicate which one of the statements below is true. The series $\sum_{n=0}^{\infty} \frac{1}{n^2+3}$
(a) $\bullet$absolutely converges (b) conditionally converges (c) diverges
Compare to the convergent p–series $\sum n = 1\frac{11}{n^2}$.

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10. (5pt) Indicate which one of the statements below is true. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2+n}}$
(a) absolutely converges (b) $\bullet$conditionally converges (c) diverges
Alternating series. Compare to p–series $\sum n = 1\frac{1}{n^{\frac{2}{3}}}$ which diverges. Terms of the alternating series decrease to 0, so converges conditionally.

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11. (5pt) Indicate which one of the statements below is true. The series $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$
(a) absolutely converges (b) conditionally converges (c) $\bullet$diverges
The Integral Test applies and we need to evaluate $\int_2^{\infty} \frac{dx}{x(\ln x)^{\frac{1}{2}}} = 2(\ln x)^{\frac{1}{2}}\Big|_2^{\infty} =$
$2 \lim_{x\to\infty} (\ln x)^{\frac{1}{2}} - 2\sqrt{\ln 2}$ which diverges since $\ln x$ goes to ∞ as x does.

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12. (5pt) Indicate which one of the statements below is true. The series

$$\sum_{n=0}^{\infty} \frac{2n+1}{(n^2+1)(n^2+2n+2)}$$

(a) has value 2 (b) $\bullet$has value 1 (c) has value 4 (d) diverges (e) has value 3
This is a telescoping series since $\frac{2n+1}{(n^2+1)(n^2+2n+2)} = \frac{1}{n^2+1} - \frac{1}{n^2+2n+2} = \frac{1}{n^2+1} - \frac{1}{(n+1)^2+1}$. Since $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$, the series sums to $\frac{1}{0^2+1}$.

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13. (5pt) Which series below is the Taylor series for the function $\ln x$ at 2?
(a) $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n}$ (b) $\sum_{n=1}^{\infty}(-1)^n \frac{(x-2)^{2n}}{(2n)!}$ (c) $\bullet \ln 2 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-2)^n}{n2^n}$
(d) $\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(x-2)^{n-3}}{\ln 2}$ (e) $\sum_{n=0}^{\infty}\frac{(x-2)^n}{n!}$
(c) is the only series which has the correct constant term.

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14. (5pt) The MacLaurin Series for $(1+x^3)^{\frac{1}{3}}$ starts out
(a) $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 \cdots$ (b) $\frac{1}{3}x^3 - \frac{1}{9}x^6 + \frac{5}{81}x^9 \cdots$ (c) $\frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 \cdots$
(d) $\sqrt[3]{1+x^3} + \cdots$ (e) $\bullet 1 + \frac{1}{3}x^3 - \frac{1}{9}x^6 + \frac{5}{81}x^9 \cdots$

Plug into the Binomial Theorem: $(1+x^3)^{\frac{1}{3}} = \sum_{n=0} \binom{\frac{1}{3}}{n}(x^3)^n = 1+\frac{1}{3}x^3+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2}x^6+\cdots =$ $1+\frac{1}{3}x^3-\frac{1}{9}x^6+\cdots$

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15. (5pt) The difference $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{2^n} - \sum_{n=1}^{4}(-1)^{n+1}\frac{n^2}{2^n}$ is

(a) •positive and less than $\frac{25}{32}$
(b) negative and greater than $-\frac{49}{1024}$
(c) negative and greater than $-\frac{25}{32}$
(d) greater than $-\frac{1}{1024}$ and less than $\frac{1}{1024}$
(e) positive and less than $\frac{49}{1024}$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{2^n}$ is an alternating series to which the Alternating Series Test applies. Hence the difference is bounded by the absolute value of the 5th term, $\frac{25}{32}$, and is positive.

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16. (5pt) The partial fraction decomposition for $\frac{x^5+4x^3-4x^2+2x-3}{x^2(x^2+1)}$ is

(a) $x+\frac{3}{x^2}+\frac{x-1}{x^2+1}$
(b) $x^2+\frac{2}{x}-\frac{5}{x^2}+\frac{x-1}{x^2+1}$
(c) $x-\frac{3}{x^2}+\frac{x-1}{x^2+1}$
(d) •$x+\frac{2}{x}-\frac{3}{x^2}+\frac{x-1}{x^2+1}$
(e) $\frac{2}{x}-\frac{5}{x^2}+\frac{x-1}{x^2+1}$

By polynomial long division, $\frac{x^5+4x^3-4x^2+2x-3}{x^2(x^2+1)} = x + \frac{3x^3-4x^2+2x-3}{x^2(x^2+1)}$ and $\frac{3x^3-4x^2+2x-3}{x^2(x^2+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}$. Hence $3x^3-4x^2+2x-3 = Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2$. Plug in $x=0$, $B=-3$. Hence $Ax(X^2+1)+(Cx+D)x^2 = 3x^3-x^2+2x$ or $A(x^2+1)+(Cx+D)x = 3x^2-x+2$. Plug in $x=0$, $A=2$. Hence $(Cx+D)x = x^2-x$ so $Cx+D = x-1$.

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17. (5pt) $\int_0^1 xe^x dx =$

(a) •1 (b) 0 (c) 3 (d) 4 (e) 2

By parts: $u = x$ $dv = e^x dx$; $du = dx$, $v = e^x$. $\int_0^1 xe^x dx = xe^x\Big|_0^1 - \int_0^1 e^x dx = xe^x\Big|_0^1 - e^x\Big|_0^1 = (e-0)-(e-1) = 1.$

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18. (5pt) The radius of convergence of $\sum_{n=0}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}x^n$ is

(a) ∞ (b) 8 (c) 3 (d) 2 (e) •1

There is a typo - the sum should start at $n=1$. Compute

$\frac{\frac{1\cdot 3\cdots(2n+1)}{2\cdot 4\cdots(2n+2)}x^{n+1}}{\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}x^n} = \frac{2n+1}{2n+2}\cdot x$. As n goes to ∞ this quantity goes to x, so the radius of Convergence is 1.

19. (5pt) $\dfrac{d\,3^x}{dx} =$

(a) $x3^{x-1}$ (b) $\ln(3^x)$ (c) $\bullet(\ln 3)3^x$ (d) $\dfrac{3^x}{\ln 3}$ (e) 3^x

$\dfrac{d\,3^x}{dx} = \dfrac{d\,(e^{\ln 3})^x}{dx} = \dfrac{d\,e^{x\ln 3}}{dx} = (\ln 3)e^{x\ln 3}$.

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20. (5pt) $\displaystyle\int_0^\infty \frac{dx}{x^2+1} =$

(a) $\frac{\pi}{3}$ (b) $\bullet\frac{\pi}{2}$ (c) diverges (d) $\arctan 4$ (e) 0

$\displaystyle\int_0^\infty \frac{dx}{x^2+1} = \lim_{x\to\infty} \arctan x - \arctan 0 = \frac{\pi}{2} - 0.$

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21. (5pt) $\displaystyle\int_0^2 \frac{dx}{x-1} =$

(a) 2 (b) $\ln 3$ (c) $\ln 2$ (d) $\bullet$diverges (e) 0

The integral is improper because $x-1$ vanishes at $x=1$. $\displaystyle\int \frac{dx}{x-1} = \ln|x-1| + C$. Since $\displaystyle\lim_{x\to 1^+} \ln|x-1| = -\infty$, the integral diverges.

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22. (5pt) Which of the functions below grows the most slowly?

(a) x^3-3x (b) $\frac{x^2}{4}$ (c) x^6+3x^5+4 (d) $\bullet x\ln x$ (e) $\ln\left(e^{x^6}\right)$

(e) is a fancy way to write the polynomial x^6. (d) grows faster than x and more slowly than x^2.

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23. (5pt) $\dfrac{d\,\operatorname{arcsec}(x^2)}{dx} =$

(a) $\dfrac{2x}{\sqrt{1-x^4}}$ (b) $\dfrac{1}{\sqrt{1-x^4}}$ (c) $\dfrac{1}{x^2\sqrt{x^4-1}}$ (d) $\dfrac{x^2}{\sqrt{1-x^4}}$ (e) $\bullet\dfrac{2}{x\sqrt{x^4-1}}$

$\dfrac{d\,\operatorname{arcsec}(x)}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$ so by the Chain Rule, $\dfrac{d\,\operatorname{arcsec}(x^2)}{dx} = \dfrac{2x}{x^2\sqrt{(x^2)^2-1}}$

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24. (5pt) Which function below is a solution to the differential equation $y' - x^2y = 0$?

(a) $y = 3 + e^{\frac{x^3}{3}}$ (b) $y = \frac{3}{2}e^{\frac{x^3}{2}}$ (c) $\bullet y = 2e^{\frac{x^3}{3}}$ (d) $y = 3e^{\frac{x^2}{2}}$ (e) $y = 2 + e^{\frac{x^2}{2}}$

This is a linear equation in standard form. The integrating factor is $v = e^{\int -x^2\,dx} = e^{\frac{-x^3}{3}}$. Hence $\left(e^{\frac{-x^3}{3}}y\right)' = 0$ so $e^{\frac{-x^3}{3}}y = C$, or $y = Ce^{\frac{x^3}{3}}$. (c) is the only function of this form among the choices.