Extra credit homework 7, due November 3

I've included solutions for each part.

Let n be a non-negative integer. The nth *Chebyshev polynomial* $T_n(x)$ is defined to be

$$T_n(x) = \cos(n \arccos x).$$

(a) What are the domain and range of this function?

Solution: Since $T_n(x)$ is the cosine of something, then its range is $[-1, 1]$. Since you need to take the arccosine of x to define $T_n(x)$, then the only allowed values for x are $[-1, 1]$, so this is the domain as well.

(b) Find simpler forms for $T_0(x)$, $T_1(x)$, $T_2(x)$, and $T_3(x)$.

Solution: I need some trig identities:

(1) $\cos(y + z) = \cos y \cos z - \sin y \sin z$
(2) $\cos(2y) = 2\cos^2 y - 1$
(3) $\sin(y + z) = \sin y \cos z + \sin z \cos y$
(4) $\sin(2y) = 2 \sin y \cos y$
(5) $\sin y \sin z = 1/2(\cos(y - z) - \cos(y + z))$
(6) $\sin y \cos z = 1/2(\sin(y - z) + \sin(y + z))$

So we get:

$$\begin{aligned}
T_0(x) &= \cos(0) = 1,\\
T_1(x) &= \cos(\arccos x) = x,\\
T_2(x) &= \cos(2 \arccos x) = 2\cos^2(\arccos x) - 1 = 2x^2 - 1,\\
T_3(x) &= \cos(3 \arccos x)\\
&= \cos(\arccos x)\cos(2 \arccos x) - \sin(\arccos x)\sin(2 \arccos x)\\
&= x(2x^2 - 1) - \sin(\arccos x) 2 \sin(\arccos x)\cos(\arccos x)\\
&= 2x^3 - x - 2\sin^2(\arccos x)\cos(\arccos x)\\
&= 2x^3 - x - 2(1 - \cos^2(\arccos x))\cos(\arccos x)\\
&= 2x^3 - x - 2(1 - x^2)x\\
&= 4x^3 - 3x.
\end{aligned}$$

(c) Show that, when n is at least 1, then

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Solution: Let $\theta = \arccos x$.

$$\begin{aligned} T_{n+1}(x) &= \cos((n+1)\theta) \\ &= \cos(\theta + n\theta) \\ &= \cos\theta\cos n\theta - \sin\theta\sin n\theta \\ &\qquad\qquad\qquad \text{(using trig identity 1)} \\ &= xT_n(x) - \frac{1}{2}(\cos((n-1)\theta) - \cos((n+1)\theta) \\ &\qquad\qquad\qquad \text{(using trig identity 5)} \\ &= xT_n(x) - \frac{1}{2}(T_{n-1}(x) - T_{n+1}(x)) \\ &= xT_n(x) - \frac{1}{2}T_{n-1}(x) + \frac{1}{2}T_{n+1}(x). \end{aligned}$$

Now solve for $T_{n+1}(x)$.

(d) Use part (c) to show that $T_n(x)$ is a polynomial of degree n.

Solution: Well, by part (b), we know that this is correct when n is 0, 1, 2, or 3. If we know that $T_n(x)$ is a polynomial of degree n, and if $T_{n-1}(x)$ is a polynomial of degree $n-1$, then by part (c), we can see that $T_{n+1}(x)$ is a polynomial of degree $n+1$. Starting with $n = 3$, we see that $T_4(x)$ is a polynomial of degree 4, $T_5(x)$ is a polynomial of degree 5, etc. (By "etc.", I really mean that you should prove this by induction.)

(e) Use (b) and (c) to express T_4, T_5, and T_6 as polynomials.

Solution: To get T_4, we apply (c) with $n = 3$:

$$\begin{aligned} T_4(x) &= 2xT_3(x) - T_2(x) \\ &= 2x(4x^3 - 3x) - (2x^2 - 1) \\ &= 8x^4 - 8x^2 - 1. \end{aligned}$$

Similarly,

$$\begin{aligned} T_5(x) &= 16x^5 - 20x^3 + 5x, \\ T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1. \end{aligned}$$

(f) What are the zeros of T_n? Where are its local maxima and minima?

Solution: $T_n(x) = \cos(n \arccos x)$, and $\cos(\theta) = 0$ when $\theta = \pm\pi/2$, $\pm 3\pi/2$, $\pm 5\pi/2$, ... So $T_n(x) = 0$ when $n \arccos x = \pm\pi/2$, $\pm 3\pi/2$, $\pm 5\pi/2$, ... In other words, $T_n(x) = 0$ when

$$\arccos x = \pm\pi/2n, \pm 3\pi/2n, \pm 5\pi/2n, \ldots, \qquad \text{or}$$
$$x = \cos(\pi/2n), \cos(3\pi/2n), \cos(5\pi/2n), \ldots$$

Since $T_n(x)$ is a polynomial of degree n, then it should only have n zeros; in fact, the terms in my list of zeros start to repeat after a while, so the zeros are actually:

$$x = \cos(\pi/2n), \cos(3\pi/2n), \cos(5\pi/2n), ..., \cos((2n-1)\pi/2n).$$

To find the local maxima and minima, we differentiate $T_n(x)$ and set the derivative equal to zero:

$$T_n'(x) = \frac{n\sin(n\arccos x)}{\sqrt{1-x^2}} = 0.$$

This is zero only when $\sin(n \arccos x) = 0$, which happens when

$$n\arccos x = 0, \pi, 2\pi, 3\pi, \ldots, \qquad \text{or}$$

$$\arccos x = 0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n} ..., \qquad \text{or}$$

$$x = \cos(0) = 1, \cos(\frac{\pi}{n}), \cos(\frac{2\pi}{n}), ..., \cos(\frac{(n-1)\pi}{n}), \cos(\frac{n\pi}{n}) = -1.$$

(g) What is the integral of $T_n(x)$ with respect to x, as x goes from -1 to 1?

Solution: Let $u = \arccos x$, so $\cos u = x$, so $-\sin u \; du = dx$.

$$\begin{aligned}
\int_{-1}^{1} \cos(n\arccos x) dx &= -\int_{\pi}^{0} \cos(nu)\sin u \; du \\
&= \frac{1}{2}\int_0^{\pi} (\sin((1-n)u) + \sin((1+n)u))du \\
&\qquad\qquad \text{(by trig identity 6)} \\
&= \frac{1}{2}\left(\frac{1}{n-1}\cos((n-1)u) - \frac{1}{n+1}\cos((n+1)u)\right)_0^{\pi} \\
&= \begin{cases} \frac{1}{2}\left(-\frac{2}{n-1} + \frac{2}{n+1}\right), & n \text{ even,} \\ 0, & n \text{ odd} \end{cases} \\
&= \begin{cases} -\frac{2}{n^2-1}, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}
\end{aligned}$$

(h) You can define Chebyshev polynomials T_a for any number a, not just a non-negative integer. What can you tell me about T_a in general? What about T_a for specific values of a (for instance, $a = \frac{1}{2}$ or $a = \frac{1}{3}$)?

Solution: This is an open-ended question; I wasn't looking for anything in particular. For example, though, $T_{\frac{1}{2}}(x) = \cos(\frac{1}{2}\arccos x)$, and you might try to simplify that using a half-angle formula.

(i) What are these Chebyshev polynomials good for, anyway?

Solution: If you want to know, then go to the library and do some research. When you get to the library, you may need to know how to spell Chebyshev:

For what it's worth, "Chebyshev" is sometimes transliterated as "Tschebyscheff" or "Čebysev" or in various other ways. See *The Thread* by Philip Davis for more information (on the transliteration problem).