Extra credit homework 13, due December 4

I've included solutions for each part.

Consider the function $f(x)$, defined to be

$$f(x) = \begin{cases} e^{-1/x^2} & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) Show that $f'(0) = 0$.

Solution: To compute $f'(0)$, you need to use the definition of derivative:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{e^{-1/h^2} - 0}{h} \\ &= \lim_{h \to 0} \frac{e^{-1/h^2}}{h} \\ &= \lim_{h \to 0} \frac{1/h}{e^{1/h^2}}. \end{aligned}$$

(Technical point: in this limit, we can assume that h is never 0, which is why I can write $f(h) = e^{-1/h^2}$.) Now use L'Hôpital's rule:

$$\begin{aligned} &= \lim_{h \to 0} \frac{-1/h^2}{-2/h^3 e^{1/h^2}} \\ &= \lim_{h \to 0} \frac{h}{2e^{1/h^2}} \\ &= 0. \end{aligned}$$

(This last limit is zero because the numerator approaches zero, while the denominator approaches ∞.)

(b) Show that $f''(0) = 0$.

Solution: This is similar. First, we need to compute $f'(x)$ when $x \neq 0$: we just apply the chain rule to $e^{-1/x^2} = e^{-x^{-2}}$: $f'(x) = \frac{2}{x^3} e^{-1/x^2}$.

So:

$$
\begin{aligned}
f''(0) &= \lim_{h\to 0} \frac{f'(h) - f'(0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{2}{h^3}e^{-1/h^2} - 0}{h} \\
&= \lim_{h\to 0} \frac{2e^{-1/h^2}}{h^4} \\
&= \lim_{h\to 0} \frac{2h^{-4}}{e^{1/h^2}} \\
&= \lim_{h\to 0} \frac{-8h^{-5}}{-2h^{-3}e^{1/h^2}} \quad \text{(L'Hôpital)} \\
&= \lim_{h\to 0} \frac{4h^{-2}}{e^{1/h^2}} \\
&= \lim_{h\to 0} \frac{-8h^{-3}}{-2h^{-3}e^{1/h^2}} \quad \text{(L'Hôpital)} \\
&= \lim_{h\to 0} \frac{4}{e^{1/h^2}} \\
&= 0.
\end{aligned}
$$

(c) Show that $f^{(n)}(0) = 0$ for as many values of n as you can. Ideally, explain why this is zero for all n.

Solution: I'm not going to give the details. One approach would be to show that

$$\lim_{h\to 0} \frac{e^{-1/h^2}}{h^k} = 0$$

for any positive integer k, and then show that $f^{(n)}(0)$ is computed as a sum of such limits.