A brief introduction to the complex numbers

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This short handout consists of 3 sections. The first gives a definition of the complex numbers, $\mathbb{C}$, and explains how to add, subtract, multiply, and divide. *This is the most important section and the* **only** *one that you are responsible for.*

The second and third sections detail some basic facts about $\mathbb{C}$. Reading through these sections will give you some feeling for the complex numbers. I refer to the fine book *Numbers* (by Ebbinghaus et al., Springer-Verlag (1990)) for a full historical development of numbers from the natural numbers on through and far beyond the complex numbers. In particular there is a whole chapter on $\mathbb{C}$ telling its many century old history, and chapters on the number π, and on the fundamental theorem of algebra.

1 Basic facts about the complex numbers

The complex numbers regarded as a set are simply the plane, $\mathbb{R}^2$. Thinking of the points in the plane as pairs of real numbers, (x, y), we see that specifying a complex number is the same as giving a pair of two real numbers. We write this pair as $x + y\sqrt{-1}$ where $\sqrt{-1}$ is a formal symbol with the multiplication $\sqrt{-1} \cdot \sqrt{-1}$ defined to be -1. This is analogous to introducing the symbol $\sqrt{2}$ with the property that $\sqrt{2} \cdot \sqrt{2} = 2$. Usually the $\sqrt{-1}$ is denoted[1] as i, which we will do from here on. We add complex numbers just as if they were

[1]for imaginary, though it is as real as most things scientists and engineers think about. Indeed most innovations in numbers were given names showing the original suspicious bent of the first users, e.g., numbers that are not fractions, e.g., numbers such as $e, \pi, \sqrt{2}$ are called irrational numbers, and roots of integers, such as $\sqrt{2}$, are called surds (short for absurd).

vectors. $(2+3i)+(4+7i)=6+10i$. We multiply complex numbers just as if they were sums of variables—except for the one relation $i^2=-1$. Thus

$$(2+3i)\cdot(5+4i)=2\cdot 5+3i\cdot 5+2\cdot 4i+3i\cdot 4i=10+15i+8i-12=-2+23i.$$

We regard the real numbers, $\mathbb{R}$,as a subset of the complex numbers, $\mathbb{C}$, by identifying a real number x with $x+0i$ which we just write x. Thus 2 is naturally the complex number $2+0i$ which we write 2, and we let 0 denote the point $0+0i$, i.e., the origin of the plane. Addition and multiplication of real numbers regarded as complex numbers gives the same answers as addition and multiplication of the real numbers thought of as real numbers. Thus $2\cdot 3=(2+0i)\cdot(3+0i)$.

Homework Problem 1 *Compute the following and write them in the form $a+bi$ with $a,b\in\mathbb{R}$.*

1. $(2+0i)\cdot(3+0i)=?\quad (2+0i)+(3+0i)=?$
2. $(2+3i)\cdot(2+3i)=?\quad (2+3i)+(2+3i)=?\quad (2+3i)-(2+3i)=?$
3. $(2+3i)\cdot(2-3i)=?$
4. $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^2=?$

If $a+bi\neq 0$ then:

$$(a+bi)\cdot\frac{a-bi}{a^2+b^2}=1.$$

Thus if $a+bi\neq 0$ then:

$$(a+bi)^{-1}=\frac{1}{a+bi}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i.$$

Homework Problem 2 *Compute and put in the form $a+bi$ the numbers:* $\frac{1}{2+3i}$, $\frac{1}{5}$, $\frac{1}{4i}$.

There are some useful operations on $\mathbb{C}$. One is conjugation. The conjugate of the complex number $2+3i$ is $2-3i$, and in general with $a,b\in\mathbb{R}$ we have that the conjugate of $a+bi$ is $a-bi$. The conjugate of a complex number $z\in\mathbb{C}$ is denoted $\overline{z}$, e.g., $\overline{2+3i}=2-3i$.

Homework Problem 3 *Compute and put in the form* $a + bi$ *the products:* $(2+3i)\cdot(\overline{2+3i})$; $(2+3i)\cdot(\overline{1+i})$.

Note that given a complex number $z = a+bi$ with $a, b \in \mathbb{R}$, $z\cdot\overline{z} = a^2+b^2$ is a nonnegative real number which is only 0 if $z = 0$. We denote the nonnegative square root of $z \cdot \overline{z}$ by $|z|$ and call it the absolute value of z.

Homework Problem 4 *Compute* $|2i|$, $|3+4i|$, $|3-4i|$, $\left|\frac{1}{3-4i}\right|$, $|-3|$.

Note for real numbers (regarded as complex numbers) this new absolute value agrees with the usual absolute value.

2 The Fundamental Theorem of Algebra

The fundamental theorem of algebra says in effect that the process of expanding number systems by adding roots of polynomials (like fractions and square roots of fractions) ends with the complex numbers.

Theorem. *Let* $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ *denote a polynomial of the* n*-th degree with coefficients* $a_i \in \mathbb{C}$*. Then there is a complex number* z_1 *such that* $p(z_1) = 0$*. By using the Euclidean algorithm for polynomials and induction it follows that there exist complex numbers* $z_1, \ldots, z_n$ *(not necessarily distinct) such that* $p(z) = a_n \prod_{i=1}^{n}(z - z_i)$.

There are many proofs of this basic result, but no purely algebraic proofs. Note that using $p(z) = z^2 - (1+2i)$ this implies that $\sqrt{1+2i}$ exists as a complex number (actually there are two square roots just as there are two square roots of 1). You might try to show that one of them is

$$\frac{\sqrt{2+2\sqrt{5}}}{2} + \frac{2}{\sqrt{2+2\sqrt{5}}}\sqrt{-1}.$$

3 The Euler identity

In calculus the Taylor series around the origin for the exponential function is computed:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \tag{1}$$

Here for a positive integer k, $k!$ denotes k-factorial, i.e., $1! = 1$, $2! = 2$, $3! = 6$, and for $k \geq 4$ we have $k! := 1 \cdot 2 \cdot 3 \cdots k$. It is a standard convention that $0! = 1$. Two other famous series are the ones for the sin and the cos:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{2}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{3}$$

These equations are obviously related. Euler (1707-1783) seems to have been the first to make the leap and rewrite e^{ix} as

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots \\ &= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] + i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] \\ &= \cos(x) + i\sin(x). \end{aligned}$$

This is an amazing relation. For example, $e^{i\pi} = \cos(\pi) + i\sin(\pi)$. Since $\cos(\pi) = 1$ and $\sin(\pi) = 0$ we get the striking relation:

$$e^{\sqrt{-1}\pi} + 1 = 0.$$

In Euler's day when -1 still made people uneasy, when π and e were known to be slightly weird numbers, and when the complex numbers were considered at best borderline; this relation, involving the 5 most common numbers in mathematics, got people quite excited.

One of the basic facts about e^x is that

$$e^{x+y} = e^x \cdot e^y. \tag{4}$$

The map, $x \to e^x$, is in fact a isomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^+, \cdot)$ where $\mathbb{R}^+$ denotes the positive real numbers. For complex numbers, the exponential map, $z \to e^z$, is not quite an isomorphism[2] since all the numbers

$$2\pi i\mathbb{Z} := \{2\pi n | n \in \mathbb{Z}\}$$

[2]The image of $\mathbb{C}$ under the exponential map is not $\mathbb{R}^+$ but $\mathbb{C}^*$, the complex numbers minus the origin. $\mathbb{C}^*$ looks like (after some stretching) an infinitely long cylinder—0 is at one end and ∞ is at the other. Geometrically the exponential map takes $\mathbb{C}$ like an infinitely long and infinitely wide sheet of aluminum foil and rolls it neatly around the cylinder.

go to 1 under exponentiation. What does equation (4) say if we use complex numbers? If we let $x, y \in \mathbb{R}$, then

$$\begin{aligned}\cos(x+y) \quad + \quad & i\sin(x+y) = e^{ix+iy} = e^{ix} \cdot e^{iy} \\ = \quad & (\cos(x) + i\sin(x))(\cos(y) + i\sin(y)) \\ = \quad & [\cos(x)\cos(y) - \sin(x)\sin(y)] + i\,[\cos(x)\sin(y) + \sin(x)\cos(y)]\,.\end{aligned}$$

Equating real and imaginary parts we get the two formulae for cosines and sines of sums of angles:

$$\begin{aligned}\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ \sin(x+y) &= \cos(x)\sin(y) + \sin(x)\cos(y).\end{aligned}$$

The Euler identity has a number of interesting consequences.

Geometrical meaning of multiplication Given a point $a + bi$ in the complex numbers, we can write $a = \rho\cos(\theta)$ and $b = \rho\sin(\theta)$ where $\rho = |a + bi| = \sqrt{a^2 + b^2}$ and $\theta := \arccos(\frac{a}{\rho})$. Thus $a + ib = \rho e^{i\theta}$. This is called the polar representation of a complex number. Thus ρ is the distance from $a + ib$ to the origin, and θ is the angle measured from the x-axis to the ray from the origin to $a + bi$. Thus if $c + di$ is another complex number, write it in polar form $c + di = \rho' e^{i\theta'}$. Then $(a + bi)(c + di) = \rho\rho' e^{i(\theta+\theta')}$. Thus

> multiplying two complex numbers is the same as multiplying their absolute values and adding the angles they make with the x-axis.

The roots of unity Given any positive integer n there is a subset of n solutions of $z^n - 1 = 0$ contained in $(\mathbb{C}^*, \cdot)$ and closed under multiplication. Let

$$\omega := e^{\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right).$$

Note that $\omega^n = e^{\frac{2\pi i}{n}n} = e^{2\pi i} = 1$. Also since n is the least positive integer, k, such that $\cos\left(\frac{2\pi}{n}k\right) = 1$ we see that n is indeed the order of this element. Geometrically ω is a point on the unit circle $\frac{360}{n}$ degrees in the counterclockwise direction from the point $(1, 0)$.

Over the complex numbers $z^n - 1$ can be factored not just as

$$(z-1) \cdot \left(\sum_{j=0}^{n-1} z^j \right),$$

but down to linear factors. Note $(\omega^j)^n - 1 = 0$ for all $1 \leq k \leq n$. Using this it follows that

$$z^n - 1 = \prod_{j=0}^{n} (z - \omega^j).$$

For example if $n = 4$ then:

$$z^4 - 1 = (z-1) \cdot (z-i) \cdot (z+1) \cdot (z+i).$$