Math 126, Test I

February 9, 1999

Multiple Choice

Problem 1.

Solution. (1) $f'(x) = \left(e^x + \ln x\right)' = e^x + \frac{1}{x} > 0$ for $x > 0$, so $f(x)$ is an increasing function on $(0, \infty)$. Using the result of Exercise 39 in §6.1 (p.456), we see that $f(x)$ must be one-to-one.
(2) $f(1) = e^1 + \ln 1 = e$ and therefore $f^{-1}(e) = f^{-1}(f(1)) = 1$ by the definition of the inverse function of $f(x)$ (p.450).
(3) By the Derivative Rule for Inverses (p.453),

$$\begin{aligned}\frac{df^{-1}}{dx}(e) &= \frac{1}{f'(f^{-1}(e))} \\ &= \frac{1}{f'(1)} \\ &= \frac{1}{e+1}. \quad \square\end{aligned}$$

Problem 2.

Solution. Let $y = \ln x$. Then $dy = \left(\ln x\right)' dx = \frac{1}{x} dx$ and

$$y = \begin{cases} 0, & \text{if } x = 1, \\ \pi, & \text{if } x = e^{\pi}. \end{cases}$$

We have

$$\begin{aligned}\int_1^{e^{\pi}} \frac{\sin(\ln x)}{x} dx &= \int_0^{\pi} \sin y \, dy \\ &= -\cos y \Big]_0^{\pi} \\ &= -\cos\pi + \cos 0 \\ &= 2. \quad \square\end{aligned}$$

Problem 3.

Solution. (1) $f'(x) = e^x + xe^x = (1+x)e^x$. Solving the equation $f'(x) = 0$, we obtain the critical point $x = -1$.
(2) $f''(x) = e^x + (1+x)e^x = (2+x)e^x$ and $f''(-1) = e^{-1} > 0$. The graph of $y = f(x)$ is concave up at the critical point $x = -1$. Thus, $x = -1$ is the local minimum point of $f(x)$. $\square$

Problem 4.

Solution. By the Evaluation of $\log_a x$ (p.478), the equation can be rewritten as

$$\frac{\ln(1+2x)}{\ln 4} = \frac{\ln 3}{\ln 2}.$$

By the Power Rule (p.460), $\ln 4 = \ln 2^2 = 2\ln 2$. So

$$\frac{1}{2}\ln(1+2x) = \ln 3$$

and

$$\ln(1+2x) = 2\ln 3 = \ln 3^2 \quad \text{(use the Power Rule again)}.$$

Since ln is one-to-one, it is sufficient to solve the equation

$$1 + 2x = 3^2 = 9.$$

The solution is $x = 4$. $\square$

Problem 5.

Solution.

$$\begin{aligned} f(x) &= \int_1^{x^2} \frac{1}{t}dt \\ &= \ln t\Big]_1^{x^2} \\ &= \ln x^2 - \ln 1 \\ &= \ln x^2 \\ &= 2\ln x \quad \text{(by the Power Rule)}. \end{aligned}$$

So $f'(x) = \big(2\ln x\big)' = \frac{2}{x}$. $\square$

Problem 6.

Solution. Note $(3)^3 + 18 = 45 \neq 0$. By the Quotient Rule (p.61),

$$\lim_{x\to 3}\frac{x^2+9}{x^3+18} = \frac{3^2+9}{3^3+18} = \frac{2}{5}. \quad \square$$

Problem 7.

Solution. By the L'Hopital's Rule,

$$\begin{aligned} \lim_{x\to 3}\frac{e^x - e^3}{\ln(4-x)} &= \lim_{x\to 3}\frac{(e^x-e^3)'}{(\ln(4-x))'} \\ &= \lim_{x\to 3}\frac{e^x}{-\frac{1}{4-x}} \quad \text{(use the Chain Rule)} \\ &= \lim_{x\to 3}(x-4)e^x \\ &= -e^3. \quad \square \end{aligned}$$

Problem 8.

Solution. Let $y = e^x$. Then $dy = e^x dx$ and

$$y = \begin{cases} 1, & \text{if } x = 0, \\ e, & \text{if } x = 1. \end{cases}$$

The definite integral can be written as

$$\begin{aligned} \int_1^e \frac{1}{y^2+1} dy &= \arctan y \Big]_1^e \\ &= \arctan e - \arctan 1 \\ &= \arctan e - \frac{\pi}{4}. \quad \square \end{aligned}$$

Problem 9.

Solution. By the Chain Rule,

$$\begin{aligned} \frac{d}{dx} \arcsin(x^2) &= \frac{1}{\sqrt{1-(x^2)^2}} (x^2)' \\ &= \frac{2x}{\sqrt{1-x^4}}. \quad \square \end{aligned}$$

Problem 10.

Solution. Let $y = -x$. Then $dy = -dx$ and

$$y = \begin{cases} 3, & \text{if } x = -3, \\ 2, & \text{if } x = -2. \end{cases}$$

We can rewrite the definite integral as

$$\begin{aligned} \int_3^2 \frac{-dy}{-y\sqrt{(-y)^2-1}} &= \int_3^2 \frac{dy}{y\sqrt{y^2-1}} \\ &= \operatorname{arcsec} y \Big]_3^2 \\ &= \operatorname{arcsec}(2) - \operatorname{arcsec}(3). \quad \square \end{aligned}$$

Partial Credit

Problem 11.

Solution. a) By the Chain Rule,

$$\begin{aligned} f'(x) &= \frac{1}{2} \cdot \frac{1}{\sqrt{1+x^2}} \cdot (x^2)' \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{1+x^2}} \cdot (2x) \\ &= \frac{x}{\sqrt{1+x^2}} > 0 \end{aligned}$$

for all $x > 0$. Thus, $f(x)$ is increasing on $(0, \infty)$ and so $f(x)$ is one-to-one (see Exercise 39 in §6.1).
b) By the Derivative for Inverses (p.453),

$$\frac{df^{-1}}{dx}(\sqrt{10}) = \frac{1}{f'(3)}$$
$$= \frac{\sqrt{10}}{3}. \quad \square$$

Problem 12.

Solution. Certainly, one can evaluate this derivative directly. But if you use the Logarithmic Differentiation to evaluate it, the computation becomes easier.

$$\begin{aligned}
\ln f(x) &= \ln \sqrt[3]{\frac{(x^2-1)^4 e^{\sin x}}{(x+1)^5}} \\
&= \frac{4}{3} \cdot \ln(x^2-1) + \frac{1}{3} \cdot \sin x \ln e - \frac{5}{3} \cdot \ln(x+1) \\
&= \frac{4}{3} \cdot \ln(x^2-1) + \frac{1}{3} \cdot \sin x - \frac{5}{3} \cdot \ln(x+1) \quad \text{(see page 460)}.
\end{aligned}$$

By the Chain Rule,

$$\begin{aligned}
\frac{f'(x)}{f(x)} &= \frac{4}{3} \cdot \frac{1}{x^2-1} \cdot (x^2)' + \frac{1}{3} \cdot \cos x - \frac{5}{3} \cdot \frac{1}{x+1} \\
&= \frac{8}{3} \cdot \frac{x}{x^2-1} \cdot + \frac{1}{3} \cdot \cos x - \frac{5}{3} \cdot \frac{1}{x+1}.
\end{aligned}$$

Therefore,

$$f'(x) = \sqrt[3]{\frac{(x^2-1)^4 e^{\sin x}}{(x+1)^5}} \left[\frac{8}{3} \cdot \frac{x}{x^2-1} \cdot + \frac{1}{3} \cdot \cos x - \frac{5}{3} \cdot \frac{1}{x+1}\right]. \quad \square$$

Problem 13.

Solution. Let $A(t)$ denote the number of specimens at the time t. Set $A_0 = A(0) = 100$ from the assumption. We have the equation

$$A(t) = A_0 e^{kt},$$

where the parameter $k > 0$. Using the assumption again, we see

$$2A_0 = A(10) = A_0 e^{10k},$$

so $e^{10k} = 2$ and

$$k = \frac{1}{10} \cdot \ln 2.$$

Now we only need to solve the following equation

$$3000 = 100 e^{kt}.$$

We have $e^{kt} = 30$ and $kt = \ln 30$. So

$$\begin{aligned}
t &= \frac{1}{k} \cdot \ln 30 \\
&= 10 \cdot \frac{\ln 30}{\ln 2}. \text{ (minutes)}. \quad \square
\end{aligned}$$

Problem 14.

Solution. Let $y = x^2$. Note $\lim_{x\to-\infty} y = \lim_{x\to-\infty} x^2 = \infty$. So we can rewrite the limit as

$$\lim_{x\to-\infty} \big(1 + \frac{x^2}{2}\big)^{\frac{1}{x^2}} = \lim_{y\to\infty} \big(1 + \frac{y}{2}\big)^{\frac{1}{y}}.$$

First, we can evaluate

$$\begin{aligned}
\lim_{y\to\infty} \ln\big(1 + \frac{y}{2}\big)^{\frac{1}{y}} &= \lim_{y\to\infty} \frac{1}{y} \ln\big(1 + \frac{y}{2}\big) \\
&= \lim_{y\to\infty} \frac{1}{1 + \frac{y}{2}} \cdot \big(\frac{y}{2}\big)' \quad \text{(use L'Hopital's Rule)} \\
&= \lim_{y\to\infty} \frac{1}{1 + \frac{y}{2}} \cdot \frac{1}{2} \\
&= 0.
\end{aligned}$$

Then by the continuity of exponential functions,

$$\begin{aligned}
\lim_{x\to-\infty} \big(1 + \frac{x^2}{2}\big)^{\frac{1}{x^2}} &= \lim_{y\to\infty} \big(1 + \frac{y}{2}\big)^{\frac{1}{y}} \\
&= \lim_{y\to\infty} e^{\ln\big(1+\frac{y}{2}\big)^{\frac{1}{y}}} \\
&= e^0 \\
&= 1. \quad \square
\end{aligned}$$

Problem 15.

Proof. By the Chain Rule,

$$\begin{aligned}
f'(t) &= \frac{1}{\sqrt{1 - \big(\frac{t^2-1}{t^2+1}\big)^2}} \cdot \big(\frac{t^2-1}{t^2+1}\big)' - \frac{2}{1+t^2} \\
&= \frac{t^2+1}{\sqrt{(t^2+1)^2 - (t^2-1)^2}} \cdot \frac{4t}{(t^2+1)^2} - \frac{2}{1+t^2} \\
&= \frac{4t}{(t^2+1)\sqrt{(t^2+1)^2 - (t^2-1)^2}} - \frac{2}{1+t^2} \\
&= \frac{4t}{(t^2+1)\sqrt{4t^2}} - \frac{2}{1+t^2} \\
&= \frac{2t}{(t^2+1)\sqrt{t^2}} - \frac{2}{1+t^2} \\
&= \frac{2}{t^2+1} \cdot \big(\frac{t}{|t|} - 1\big). \qquad (\because \;\; \sqrt{t^2} = |t| \;)
\end{aligned}$$

Therefore,

$$f'(t) = \begin{cases} 0, & \text{if } \; t > 0, \\ -\frac{4}{t^2+1}, & \text{if } \; t < 0. \end{cases}$$

a) Since $f'(t) = 0$ for all $t \in (0, \infty)$, $f(t)$ is constant for $t \geqslant 0$;

b) Since $f'(t)$ is not identically equal to zero, $f(t)$ is not constant for $t < 0$. $\square$