Math 126, Test II

March 18, 1999

Multiple Choice

Problem 1. $y = \sqrt{x^2 + C}$

Solution. Actually, this is not an initial value problem, for you don't know the initial condition of the solution to the differential equation. So you have to find the general solution to this equation.

This equation is separable. It can be rewritten as

$$y\,dy = x\,dx. \tag{1}$$

Integrating the equation (1), one has

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1. \tag{2}$$

Multiplying the equation (2) by 2 and setting $C = 2C_1$,

$$y^2 = x^2 + C,$$

or

$$y = \begin{cases} \sqrt{x^2 + C}, & \text{if } y > 0, \\ -\sqrt{x^2 + C}, & \text{if } y < 0. \end{cases} \qquad \square$$

Problem 2. $\frac{1}{8}$

Solution. Use substitution. Let $x = \sin t$. Then $dx = \cos t\,dt$.

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^7 t \, \cos t \, dt &= \int_0^1 x^7\,dx \\ &= \frac{1}{8}x^8\Big]_0^1 \\ &= \frac{1}{8}. \qquad \square \end{aligned}$$

Problem 3. converges to 2

Solution. By the definition of improper integral with infinite integration interval,

$$\begin{aligned} \int_1^\infty \frac{1}{x^{3/2}}\,dx &= \lim_{A\to\infty} \int_1^A \frac{1}{x^{3/2}}\,dx \\ &= \lim_{A\to\infty} (-2)x^{-1/2}\Big]_1^A \\ &= \lim_{A\to\infty} (-2)\left[A^{-1/2} - 1\right] \\ &= 2. \qquad \square \end{aligned}$$

Problem 4. $\frac{3}{x+1} + \frac{2}{x+3}$

Solution. Let
$$\frac{5x+11}{x^2+4x+3} = \frac{A}{x+1} + \frac{B}{x+3}.$$
Clear the fractions.
$$5x + 11 = A(x+3) + B(x+1) = (A+B)x + (3A+B).$$
Compare the coefficients of the like-terms.
$$\begin{cases} A + B = 5, \\ 3A + B = 11. \end{cases}$$
So $A = 3$ and $B = 2$ and
$$\frac{5x+11}{x^2+4x+3} = \frac{3}{x+1} + \frac{2}{x+3}. \qquad \square$$

Problem 5. $\frac{1}{120}$

Solution. Compute $a_2, \cdots, a_5$ inductively.
$a_2 = \frac{a_1}{2} = \frac{1}{2}$;
$a_3 = \frac{a_2}{3} = \frac{1}{6}$;
$a_4 = \frac{a_3}{4} = \frac{1}{24}$;
$a_5 = \frac{a_4}{5} = \frac{1}{120}$. $\square$

Problem 6. e^{3x}

Solution.
$$\begin{aligned}(\sinh x + \cosh x)^3 &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)^3 \\ &= (e^x)^3 \\ &= e^{3x}. \qquad \square\end{aligned}$$

Problem 7. converges to $\frac{\pi}{2}$

Solution. Note that 3 is the unique singular point of the integrand.
$$\begin{aligned}\int_0^3 \frac{dx}{\sqrt{9-x^2}} &= \lim_{a\to 3}\int_0^a \frac{dx}{\sqrt{9-x^2}} \\ &= \lim_{a\to 3} \arcsin \frac{x}{3}\Big]_0^a \\ &= \lim_{a\to 3} \arcsin \frac{a}{3} - \arcsin 0 \\ &= \frac{\pi}{2}. \qquad \square\end{aligned}$$

Problem 8. $2 \ln 2 - \frac{3}{4}$

Solution. Use partial integration.

$$\begin{aligned}
\int_1^2 x \ln x \, dx &= \frac{1}{2} \int_1^2 \ln x \, d(x^2) \\
&= \frac{1}{2} \big[x^2 \ln x \Big]_1^2 - \int_1^2 x^2 (\ln x)' \, dx \big] \\
&= \frac{1}{2} \big[4 \ln 2 - \int_1^2 x \, dx \big] \\
&= \frac{1}{2} \big[4 \ln 2 - \frac{1}{2} x^2 \Big]_1^2 \big] \\
&= 2 \ln 2 - \frac{3}{4}. \qquad \square
\end{aligned}$$

Problem 9. $\frac{1}{4} e^{2x}(2x - 1) + C$

Solution. Use partial integration again.

$$\begin{aligned}
\int x e^{2x} \, dx &= \frac{1}{2} \int x \, d(e^{2x}) \\
&= \frac{1}{2} \big[x e^{2x} - \int e^{2x} \, dx \big] \\
&= \frac{1}{2} (x e^{2x} - \frac{1}{2} e^{2x}) + C \\
&= \frac{1}{4} e^{2x} (2x - 1) + C. \qquad \square
\end{aligned}$$

Problem 10. $\frac{\pi}{4}$

Solution. Complete the square.

$$\begin{aligned}
\int_1^2 \frac{dx}{x^2 - 2x + 2} &= \int_1^2 \frac{dx}{1 + (x-1)^2} \\
&= \arctan(x - 1) \Big]_1^2 \\
&= \arctan 1 - \arctan 0 \\
&= \frac{\pi}{4}. \qquad \square
\end{aligned}$$

Partial Credit

Problem 11.

Solution. Use trigonometric substitution. Let $x = \tan t$. Then $dx = \sec^2 t\, dt$ and $t = \arctan x$.

$$\begin{aligned}
\int \frac{dx}{(\sqrt{1+x^2})^3} &= \int \frac{\sec^2 t}{\sec^3 t}\, dt \\
&= \int \cos t\, dt \\
&= \sin t + C \\
&= \sin(\arctan x) + C.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{1}{\sin^2 u} &= \csc^2 u \\
&= 1 + \cot^2 u \\
&= 1 + \frac{1}{\tan^2 u} \\
&= \frac{1+\tan^2 u}{\tan^2 u}.
\end{aligned}$$

We see

$$\sin u = \frac{\tan u}{\sqrt{1+\tan^2 u}}.$$

Set $u = \arctan x$. Then $\tan u = x$ and

$$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}.$$

Thus,

$$\int \frac{dx}{(\sqrt{1+x^2})^3} = \frac{x}{\sqrt{1+x^2}} + C. \qquad \square$$

Problem 12.

Solution. We will use **Theorem 4** (p. 533) to solve this initial value problem.

First, we write the differential equation in its standard form

$$\frac{dy}{dx} + \frac{1}{x} y = \frac{1}{\sqrt{x}}.$$

Set $P(x) = \frac{1}{x}$ and $Q(x) = \frac{1}{\sqrt{x}}$.

$$v(x) = e^{\int \frac{1}{x}\, dx} = e^{\ln x} = x.$$

$$\begin{aligned}
y &= \frac{1}{v(x)} \int v(x) Q(x)\, dx \\
&= \frac{1}{x} \int \sqrt{x}\, dx \\
&= \frac{1}{x}\big(\frac{2}{3} x^{3/2} + C\big).
\end{aligned}$$

Then we need determine C. From the initial condition,

$$1 = y(1) = \frac{2}{3} + C.$$

So $C = \frac{1}{3}$ and the solution is

$$y = \frac{1}{3x}(2x^{3/2} + 1). \qquad \square$$

Problem 13.

Solution. Use partial fractions method.

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^2 + 3x + 2} &= \lim_{A\to\infty} \int_1^A \frac{dx}{x^2 + 3x + 2} \\
&= \lim_{A\to\infty} \int_1^A \frac{dx}{(x+1)(x+2)} \\
&= \lim_{A\to\infty} \int_1^A \left(\frac{1}{x+1} - \frac{1}{x+2}\right) \\
&= \lim_{A\to\infty} \big(\ln(x+1) - \ln(x+2)\big)\Big]_1^A \\
&= \lim_{A\to\infty} \ln \frac{x+1}{x+2}\Big]_1^A \\
&= \lim_{A\to\infty} \left(\ln \frac{A+1}{A+2} - \ln \frac{1+1}{1+2}\right) \\
&= \ln 1 - \ln \frac{2}{3} \\
&= \ln \frac{3}{2}. \qquad \square
\end{aligned}$$

Problem 14.

Solution.

$$\begin{aligned}
\int \arctan x\, dx &= x \arctan x - \int x(\arctan x)'\, dx \\
&= x \arctan x - \int \frac{x}{x^2+1}\, dx \\
&= x \arctan x - \frac{1}{2} \int \frac{1}{x^2+1} d(x^2) \\
&= x \arctan x - \frac{1}{2} \ln(x^2+1) + C. \qquad \square
\end{aligned}$$

Problem 15.

Solution. a)

$$\begin{aligned} a_{n+1} - a_n &= \big[1 - \big(\frac{2}{3}\big)^{n+1}\big] - \big[1 - \big(\frac{2}{3}\big)^{n}\big] \\ &= \big(\frac{2}{3}\big)^{n} - \big(\frac{2}{3}\big)^{n+1} \\ &= \big(\frac{2}{3}\big)^{n}\big(1 - \frac{2}{3}\big) \\ &= \frac{1}{3}\big(\frac{2}{3}\big)^{n} \\ &\geqslant 0. \end{aligned}$$

So $a_n \leqslant a_{n+1}$.

b)

$$a_n = 1 - \big(\frac{2}{3}\big)^{n} < 1.$$

Thus, 1 is an upper bound.

c)

$$\lim_{n\to\infty} a_n = 1. \qquad \square$$