Math 126, Test III

April 20, 1999

Multiple Choice

Problem 1. 1) conditionally converges, 2) diverges

Solution. 1) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an alternating series, so it converges by the Leibniz's Theorem.

The corresponding series of absolute values is equal to $\sum_{n=1}^{\infty} \frac{1}{n}$. It diverges because it is the harmonic series.

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conditionally converges.

2) Note

$$\lim_{n\to\infty} \frac{\sqrt{n^3-1}}{3n-1} = \infty.$$

By the n-th Term Test, $\sum_{n=1}^{\infty} \frac{\sqrt{n^3-1}}{3n-1}$ diverges.

Problem 2. ∞

Solution. Let $A_n = \frac{(x-5)^n}{n^n}$; i.e, A_n denotes the n-th term of the power series.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| &= \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}}|x-5| \\
&= \lim_{n\to\infty} \frac{1}{n+1}\frac{n^n}{(n+1)^n}|x-5| \\
&= \lim_{n\to\infty} \frac{1}{n+1}\frac{1}{\left(1+\frac{1}{n}\right)^n}|x-5| \\
&= \lim_{n\to\infty} \frac{1}{n+1} \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}|x-5| \\
&= 0 \cdot \frac{1}{e} \cdot |x-5| \\
&= 0
\end{aligned}$$

for all x. By the Ratio Test, this power series converges absolutely for all x and therefore the convergence radius $R = \infty$.

Problem 3. $\sum_{n=0}^{\infty}(n+1)2^n x^n$

Solution. Using geometric series, one has the following expression

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty} 2^n x^n \quad \text{for} \quad |2x|<1, \quad \text{or} \quad |x|<\frac{1}{2}.$$

By the Term-by-term Differentiation Theorem,

$$(1)\qquad \begin{aligned}\Big(\frac{1}{1-2x}\Big)' &= \sum_{n=0}^{\infty}(2^n x^n)' \\ &= \sum_{n=0}^{\infty} n2^n x^{n-1} \\ &= \sum_{n=1}^{\infty}(n+1)2^{n+1}x^n \quad \text{for} \quad |x| < \frac{1}{2}.\end{aligned}$$

On the other hand,

$$(2)\qquad \Big(\frac{1}{1-2x}\Big)' = \frac{2}{(1-2x)^2}.$$

From (1) and (2), one obtains

$$\begin{aligned}\frac{1}{(1-2x)^2} &= \frac{1}{2}\Big(\frac{1}{1-2x}\Big)' \\ &= \frac{1}{2}\sum_{n=1}^{\infty}(n+1)2^{n+1}x^n \\ &= \sum_{n=1}^{\infty}(n+1)2^n x^n \quad \text{for} \quad |x| < \frac{1}{2}.\end{aligned}$$

Problem 4. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{4k+2}$

Solution. Note

$$\sin(y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} y^{2k+1}.$$

Replacing y by $2x$, we get

$$\begin{aligned}\sin(x^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(x^2)^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{4k+2}.\end{aligned}$$

Problem 5. $x + x^2 + \frac{1}{3}x^3$

Solution. Assume that $e^x = \sum_{n=0}^{\infty} a_n x^n$, $\sin x = \sum_{n=0}^{\infty} b_n x^n$ and $e^x \sin x = \sum_{n=0}^{\infty} c_n x^n$.

We have the following Maclaurin series

$$\begin{aligned}e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.\end{aligned}$$

Both of these series are absolutely convergent for all x.

Computing the first four terms of the two Maclaurin series explicitly and comparing the coefficients of the like-terms, we see

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{6}.$$

$$b_0 = 0, \quad b_1 = 1, \quad b_2 = 0, \quad b_4 = -\frac{1}{6}.$$

By the Series Multiplication Theorem for Power Series (page 670),

$$\begin{aligned} c_0 &= a_0 b_0 = 0, \\ c_1 &= a_0 b_1 + a_1 b_0 = 1, \\ c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 = 1, \\ c_3 &= a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = \frac{1}{3}. \end{aligned}$$

Hence, the first three nonzero terms of the Maclaurin series expansion of $e^x \sin x$ are

$$x + x^2 + \frac{1}{3}x^3.$$

Problem 6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Solution. a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^3+1}{n^2+2}$ diverges because its n-th term does not go to 0.
b) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, since it is the harmonic series.
c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, since it is an alternating series. But the corresponding series of absolute values is the harmonic series and so diverges.
d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely by the p-series Test.
e) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges absolutely by the Direct Comparison Test and the p-series Test.

Problem 7. 1

Solution. The m-th partial sum is

$$\begin{aligned} s_m &= \sum_{n=0}^{\infty} \left(e^{-n} - e^{-(n+1)}\right) \\ &= (1 - e^{-1}) + (e^{-1} - e^{-2}) + (e^{-2} - e^{-3}) + \cdots + (e^{-(m-1)} - e^{-m}) \\ &= 1 - e^{-m}. \end{aligned}$$

So $\sum_{n=0}^{\infty} \left(e^{-n} - e^{-(n+1)}\right) = \lim_{m\to\infty}(1 - e^{-m}) = 1$.

Problem 8. 1) and 2)

Solution. By the p-series Test, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Compare the term $\frac{n^2-4n+10}{n^4+3n^2+10}$ with $\frac{1}{n^2}$. By the Limit Comparison Test and the p-series Test, $\sum_{n=1}^{\infty} \frac{n^2-4n+10}{n^4+3n^2+10}$ converges.

Problem 9. $\frac{1}{6}$

Solution. $\cos(x)$ has the following absolutely convergent Maclaurin series expansion

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

$$\begin{aligned}
4(\cos(x) - 1) + 2x^2 &= 4(1 - \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} - 1) + 2x^2 \\
&= 4(-\frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}) + 2x^2 \\
&= -2x^2 + 4\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + 2x^2 \\
&= 4\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{x\to 0} \frac{4(\cos(x) - 1) + 2x^2}{x^4} &= \lim_{x\to 0} 4\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{2n}}{x^4} \\
&= \lim_{x\to 0} 4\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} x^{2(n-1)} \\
&= \lim_{x\to 0} \left[4\frac{1}{4!} + 4\sum_{n=3}^{\infty} \frac{(-1)^n}{(2n)!} x^{2(n-1)}\right] \\
&= \frac{4}{24} \\
&= \frac{1}{6}.
\end{aligned}$$

Problem 10. 0

Solution. Assume that $y(x) = \sum_{n=0}^{\infty} c_n x^n$ sloves the initial value problem and that this power series converges absolutely. By the initial condition, $c_0 = y(0) = -2$.

Plug the series $\sum_{n=0}^{\infty} c_n x^n$ into the series and use the Term-by-term Differentiation Theorem.

$$\begin{aligned}
\Big(\sum_{n=0}^{\infty} c_n x^n\Big)' &= \sum_{n=0}^{\infty} c_n x^n + x^2, \\
\sum_{n=0}^{\infty} (c_n x^n)' &= \sum_{n=0}^{\infty} c_n x^n + x^2, \\
\sum_{n=0}^{\infty} c_n n x^{n-1} &= \sum_{n=0}^{\infty} c_n x^n + x^2, \\
\sum_{n=1}^{\infty} c_n n x^{n-1} &= \sum_{n=0}^{\infty} c_n x^n + x^2, \\
\sum_{n=0}^{\infty} c_{n+1}(n+1) x^n &= \sum_{n=0}^{\infty} c_n x^n + x^2.
\end{aligned}$$

$$\sum_{n=0}^{\infty} c_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} c_n x^n - x^2 = 0.$$

But

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} c_n x^n - x^2 \\
&= \sum_{n=0}^{\infty}[(n+1)c_{n+1} - c_n]x^n - x^2 \\
&= (c_1 - c_0) + (2c_2 - c_1)x + (3c_3 - c_2)x^2 + \sum_{n=3}^{\infty}[(n+1)c_{n+1} - c_n]x^n - x^2 \\
&= (c_1 - c_0) + (2c_2 - c_1)x + (3c_3 - c_2 - 1)x^2 + \sum_{n=3}^{\infty}[(n+1)c_{n+1} - c_n]x^n.
\end{aligned}$$

Comparing the coefficients of the like-terms,

$$\begin{cases} c_1 - c_0 = 0, \\ 2c_2 - c_1 = 0, \\ 3c_3 - c_2 - 1 = 0. \end{cases}$$

So

$$\begin{aligned}
c_1 &= c_0 = -2, \\
c_2 &= \frac{1}{2}c_1 = -1, \\
c_3 &= \frac{1}{3}(c_2 + 1) = 0.
\end{aligned}$$

The third order term is 0.

Partial Credit

Problem 11.

Solution. Denote the n-th term by $A_n = (-1)^{n+1}\frac{(x-1)^n}{n}$. Then

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\frac{n}{n+1}|x-1| = |x-1|.$$

By the ratio test, the power series converges absolutely for $|x-1|<1$; i.e, $0<x<2$ and the power series diverges for $|x-1|>1$; i.e., $x<0$ and $x>2$.

When $x=0$, the series is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$. It diverges, for $\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series.

When $x=2$, the series is $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n}$. It converges, for it is an alternating series but it is not absolutely convergent.

Hence the convergence interval is $0<x\leqslant 2$.

Problem 12.

Solution. Note $e^y = \sum\frac{y^n}{n!}$. e^{-t^2} has the absolutely convergent Maclaurin series expansion

$$e^{-t^2} = \sum_{n=0}^{\infty}\frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{n!}.$$

By the Term-by-term Integration Theorem,

$$\begin{aligned}\int_0^x e^{-t^2}\,dt &= \int_0^x\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{n!}\,dt\\ &= \sum_{n=0}^{\infty}\int_0^x(-1)^n\frac{t^{2n}}{n!}\,dt\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^x t^{2n}\,dt\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{2n+1}t^{2n+1}\Big]_0^x\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{2n+1}x^{2n+1}.\end{aligned}$$

Problem 13.

Solution. Set $f(x) = \frac{1}{(\arctan x)^2(1+x^2)}$ for $x>1$.
(1) $f(n) = \frac{1}{(\arctan n)^2(1+n^2)}$.
(2) Since $\arctan x$ and $1+x^2$ are continuous functions, so is $f(x)$.
(3) Since both $\arctan x$ and $1+x^2$ are increasing, $f(x)$ is decreasing.

(4)

$$\begin{aligned}\lim_{x\to\infty}\frac{1}{(\arctan x)^2(1+x^2)} &= \lim_{x\to\infty}\frac{1}{(\arctan x)^2}\lim_{x\to\infty}\frac{1}{1+x^2}\\ &= \frac{1}{\left(\frac{\pi}{2}\right)^2}\cdot 0\\ &= 0.\end{aligned}$$

$$\begin{aligned}\int_1^\infty f(x)\,dx &= \int_1^\infty \frac{dx}{(\arctan x)^2(1+x^2)}\\ &= \int_1^\infty \frac{d(\arctan x)}{(\arctan x)^2}\\ &= -\frac{1}{\arctan x}\Big]_1^\infty\\ &= -\frac{1}{\frac{\pi}{2}}+\frac{1}{\frac{\pi}{4}}\\ &= \frac{2}{\pi}.\end{aligned}$$

By the Integral Test, this series converges.

Problem 14.

Solution. Note

(1) $\frac{1}{n\ln^2 n} > 0$;
(2) $\frac{1}{n\ln^2 n}$ is decreasing;
(3) $\lim_{n\to\infty}\frac{1}{n\ln^2 n} = 0$.

$\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln^2 n}$ is an alternating series so it converges.

By the Alternating Series Estimation Theorem (page 657),

$$\Big|\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln^2 n}-\sum_{n=2}^{10}\frac{(-1)^n}{n\ln^2 n}\Big| \leqslant \Big|\frac{(-1)^{11}}{11\ln^2 11}\Big| = \frac{1}{11\ln^2 11}.$$

The difference has the same sign as $\frac{(-1)^{11}}{11\ln^2 11}$. So the difference is negative.

Problem 15.

Solution.

$$\begin{aligned} f(x) &= \cos x, & f(\pi) &= -1;\\ f'(x) &= -\sin x, & f'(\pi) &= 0;\\ f''(x) &= -\cos x, & f''(\pi) &= 1;\\ f'''(x) &= \sin x, & f'''(\pi) &= 0;\\ f^{(4)}(x) &= \cos x, & f^{(4)}(\pi) &= -1.\end{aligned}$$

The order 4 Taylor polynomial is

$$\begin{aligned}& f(\pi)+f'(\pi)(x-\pi)+\frac{f''(\pi)}{2!}(x-\pi)^2+\frac{f'''(\pi)}{3!}(x-\pi)^3+\frac{f^{(4)}(\pi)}{4!}(x-\pi)^4\\ &= -1+\frac{1}{2}(x-\pi)^2-\frac{1}{24}(x-\pi)^4.\end{aligned}$$