Math 126: Calculus II, Exam II Solutions

1 a) Using integration by parts with $u = x$, $dv = e^{2x}dx$, $du = dx$, $v = (1/2)e^{2x}$, gives $\int xe^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$

b) Using the trig substitution $x = \tan(\theta)$, $dx = \sec^2(\theta)d\theta$, $\sqrt{x^2+1} = \sec(\theta)$, gives $\int \frac{\sqrt{x^2+1}}{x^4}\,dx = \int \frac{\sec^3(\theta)}{\tan^4(\theta)}\,d\theta = \int \frac{\cos(\theta)}{\sin^4(\theta)}\,d\theta$. Substituting $u = \sin(\theta)$, $du = \cos(\theta)d\theta$ gives $\int u^{-4}du = -\frac{1}{3\sin^3(\theta)} + C = -\frac{(x^2+1)^{3/2}}{3x^3} + C$.

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2. The partial fraction decomposition of $\frac{x+2}{x^2(x^2+1)}$ looks like $\frac{x+2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$. Multiplying by $x^2(x^2+1)$ gives $x+2 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 = (A+C)x^3 + (B+D)x^2 + Ax + B$. Therefore, $A = 1$, $B = 2$, $A + C = 0$, and $B + D = 0$, so $C = -1$, $D = -2$.

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3. $\sum_{n=1}^{\infty} \frac{3^n + 2(-1)^{n+1}}{4^n} = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n - 2\sum_{n=1}^{\infty} \left(\frac{-1}{4}\right)^n = \frac{3/4}{1-3/4} - 2\frac{-1/4}{1-(-1/4)} = 3 + 2\frac{1}{5} = \frac{17}{5}$

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4. a) $a_n = \frac{(-1)^n n}{n+1}$ diverges because one subsequence converges to 1, $a_{2k} = \frac{2k}{2k+1} \to 1$, and another subsequence converges to -1, $a_{2k+1} = \frac{-(2k+1)}{2k+2} \to -1$.

b) $a_n = n\,\sin(1/n) = \frac{\sin(1/n)}{1/n}$, so applying L'Hopital's Rule gives $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{n\to\infty} \frac{\cos(1/n)(-1/n^2)}{-1/n^2} = \cos(0) = 1$.

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5. Consider $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$.

a) The 4^{th} partial sum is $s_4 = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \ln\left(\frac{4}{5}\right) = \ln(1) - \ln(2) + \ln(2) - \ln(3) + \ln(3) - \ln(4) + \ln(4) - \ln(5) = -\ln(5)$

b) The series is telescoping, since $\ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$, so the n^{th} partial sum is $s_n = \ln(1) - \ln(n+1)$ (as in part a). Therefore, the limit of s_n is $\lim_{n\to\infty} s_n = \lim_{n\to\infty} -\ln(n) = -\infty$ and the series diverges.

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6. a) $\displaystyle\sum_{n=2}^{\infty}\Big(1-\frac{1}{n}\Big)^n$ diverges because the n^{th} term does not approach 0:
$\displaystyle\lim_{n\to\infty}\Big(1-\frac{1}{n}\Big)^n = e^{-1}\neq 0.$

b) $\displaystyle\sum_{n=2}^{\infty}\frac{\sqrt{n}}{n^2-1}$ converges by the Limit Comparison Test: We compare the series to $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^{3/2}}$ which converges because it is a p-series with $p=3/2>1$. Taking the limit of the quotient of the terms of these series gives $\displaystyle\lim_{n\to\infty}\frac{\sqrt{n}/(n^2-1)}{1/n^{3/2}} = \lim_{n\to\infty}\frac{n^2}{n^2-1} = \lim_{n\to\infty}\frac{1}{1-1/n^2} = 1.$

c) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^3}$ converges by the Integral Test: Using the substitution $u=\ln(u)$, $du=\dfrac{1}{u}du$, gives $\displaystyle\int_2^{\infty}\frac{1}{x(\ln(x))^3}\,dx = \int_{\ln(2)}^{\infty}\frac{1}{u^3}\,du = \lim_{b\to\infty} -\frac{1}{2u^2}\Big|_{\ln(2)}^{b} = \lim_{b\to\infty} -\frac{1}{2b^2}+\frac{1}{2(\ln(2))^2}$
$\displaystyle = \frac{1}{2(\ln(2))^2} < \infty.$

d) $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{n^3}$ diverges by the Ratio Test:
$\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{5^{n+1}}{(n+1)^3}\frac{n^3}{5^n} = \lim_{n\to\infty}5\Big(\frac{n}{n+1}\Big)^3 = 5>1.$

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7. a) The series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{(2n-1)!}$ converges because it is alternating, $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}u_n$, with u_n positive and decreasing to 0: $\displaystyle\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(2(n+1)-1)!}\frac{(2n-1)!}{n!} = \frac{n+1}{(2n+1)(2n)} < 1$ and $\displaystyle\lim_{n\to\infty}u_n = \lim_{n\to\infty}\frac{n!}{(2n-1)!} = \lim_{n\to\infty}\frac{1}{(2n-1)(2n-2)\cdots(n+1)} = 0.$

b) Since the error of approximating an alternating series by the n^{th} partial sum s_n is less than u_{n+1}, we can guarantee an error less than 10^{-2} by choosing n such that $u_{n+1}<10^{-2}$. Note that $n=2$ is not good enough: $u_3=\dfrac{3!}{5!}=\dfrac{1}{20}=.05>10^{-2}$; but $n=3$ works: $u_4=\dfrac{4!}{7!}=\dfrac{1}{7\cdot 6\cdot 5}=\dfrac{1}{210}=.00476<10^{-2}$. Therefore, $s_3=1-\dfrac{2!}{3!}+\dfrac{3!}{5!}=1-\dfrac{1}{3}+\dfrac{1}{20}=\dfrac{43}{60}=.716$ approximates the series with an error less than 10^{-2}.
