Math 126: Calculus II, Exam I Solutions

1. a) Let $f(x) = x \ln x$. Find the value of $(f^{-1})'(2e^2)$.
$f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$.
Since $f(e^2) = 2e^2$, $(f^{-1})'(2e^2) = \frac{1}{f'(e^2)} = \frac{1}{\ln(e^2)+1} = \frac{1}{2+1} = \frac{1}{3}$.
b) $\frac{d}{dx} \cos^{-1}(x^{-1}) = -\frac{1}{\sqrt{1-(x^{-1})^2}}\left(-x^{-2}\right) = \frac{1}{x^2\sqrt{1-(1/x^2)}} = \frac{1}{|x|\sqrt{x^2-1}}$.
c) $\frac{d}{dx} x^{\cosh(x)} = \frac{d}{dx} e^{\cosh(x)\ln(x)} = e^{\cosh(x)\ln(x)}(\sinh(x)\ln(x) + \cosh(x)(1/x))$
$= x^{\cosh(x)}(\sinh(x)\ln(x) + \cosh(x)/x)$.

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2. a) $\int e^x \sin(1+e^x)\,dx$. Let $u = 1+e^x$, $du = e^x\,dx$. Then $\int e^x \sin(1+e^x)\,dx = \int \sin(u)\,du = -\cos(u) + C = -\cos(1+e^x) + C$.
b) $\int_0^1 \frac{x}{1+x^4}\,dx$. Let $u = x^2$, $du = 2x\,dx$, or $x\,dx = (1/2)du$, and the new limits are from $u = 0^2$ to $u = 1^2$. Thus $\int_0^1 \frac{x}{1+x^4}\,dx = \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}(u)\Big|_0^1 = \frac{1}{2}[\tan^{-1}(1) - \tan^{-1}(0)] = (1/2)[\pi/4 - 0] = \pi/8$.
c) $\int \frac{x}{x+1}\,dx$. First divide: $\frac{x}{x+1} = 1 - \frac{1}{x+1}$. Then $\int \frac{x}{x+1}\,dx = \int 1 - \frac{1}{x+1}\,dx = x - \ln|x+1| + C$.

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3. a) $\lim_{x\to 0} \frac{\sin^{-1}(x)}{x}$. The numerator and denominator both approach 0, so by L'Hôpital's Rule the limit equals $\lim_{x\to 0} \frac{1/\sqrt{1-x^2}}{1} = 1$.
b) $\lim_{x\to 0^+} e^{-1/x}\ln(x)$. Note that $e^{-1/x} \to 0$ and $\ln(x) \to \infty$. Rewrite the limit as $\lim_{x\to 0^+} \frac{\ln(x)}{e^{1/x}}$ so both the numerator and denominator approach infinity, and then apply L'Hôpital's Rule: $\lim_{x\to 0^+} \frac{\ln(x)}{e^{1/x}} = \lim_{x\to 0^+} \frac{1/x}{e^{1/x}(-1/x^2)} = \lim_{x\to 0^+} -\frac{x}{e^{1/x}}$. The numerator approaches 0 and the denominator approaches infinity, so the limit is 0.

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4. Decide whether $g(x)$ grows faster than, slower than, or at the same rate as $f(x)$.

a) $f(x) = \dfrac{1}{x}$ and $g(x) = \sin(\tan^{-1}(x))$. Since $\tan^{-1}(x) \to \pi/2$ and $1/x \to 0$ as $x \to \infty$, $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim_{x\to\infty} \dfrac{1/x}{\sin(\tan^{-1}(x))} = \dfrac{1}{\sin(\pi/2)} = \dfrac{0}{1} = 0$. (Note that L'Hôpital's Rule does not apply.) Therefore, $g(x)$ grows faster than $f(x)$.

b) $f(x) = \sinh(\ln(x))$ and $g(x) = x$. Since $\sinh(\ln(x)) = (e^{\ln(x)} - e^{-\ln(x)})/2 = (x - 1/x)/2$, $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim_{x\to\infty} \dfrac{(x-1/x)/2}{x} = \lim_{x\to\infty} (1 - 1/x^2)/2 = 1/2$. (note that repeatedly applying L'Hôpital's Rule to $\sinh(\ln(x))/x$ brings you back to the beginning.) Therefore, $g(x)$ grows at the same rate as $f(x)$.

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5. Warfarin is a drug used as an anticoagulant. After administration of the drug ends, the quantity remaining in a patient's body decreases at a rate proportional to the quantity remaining. The half-life of Warfarin in the body is 37 hours. How many days does it take for the drug level in the body to be reduced to 10% of the original level.

Solution. Let $y(t)$ be the quantity remaining after t hours and let $y_0 = y(0)$ be the (unknown) initial amount. We must solve for the time t_1 such that $y(t_1) = 0.1y_0$. Since $y(t)$ decays exponentially, $y(t) = y_0e^{kt}$. The half life is 37 hours which means that $(1/2)y_0 = y(37) = y_0e^{37k}$. Solving for k gives $k = \ln(1/2)/37$. Therefore, $y(t) = y_0e^{\ln(1/2)t/37} = y_0\left(\frac{1}{2}\right)^{t/37}$. We now solve for t_1: $0.1y_0 = y(t_1) = y_0\left(\frac{1}{2}\right)^{t_1/37}$. Taking logarithms gives $\ln(0.1) = (t_1/37)\ln(0.5)$ so $t_1 = 37\ln(0.1))/\ln(0.5) = 122.9$ hours. The number of days is $t_1/24 = 122.9/24 = 5.1$.

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6. a) Solve the differential equation $\quad xy' + 2y = x$.
The standard form is $y' + \dfrac{2}{x}y = 1$, so $p(x) = \dfrac{2}{x}$ and $q(x) = 1$. Multiplying the equation by $v(x) = e^{\int 2/x\,dx} = e^{2\ln(x)} = x^2$ gives $x^2y' + 2xy = x^2$. The left-hand side is the derivative of x^2y' so $\dfrac{d}{dx}(x^2y) = x^2$. Integrating gives, $x^2y = \dfrac{1}{3}x^3 + C$, or $y = \dfrac{x}{3} + \dfrac{C}{x^2}$.

b) Solve the initial value problem $\quad x^2y' = y, \quad y(1) = 1$.
Separate the variables, $\dfrac{dy}{y} = \dfrac{dx}{x^2}$, and integrate, $\displaystyle\int \frac{dy}{y} = \int \frac{dx}{x^2}$. This gives $\ln(y) = -\dfrac{1}{x} + C$. Plug in the initial condition $y = 1$ when $x = 1$: $\ln(1) = -1 + C$, so $C = 1$. Exponentiating gives $y = e^{1-1/x}$.