

Math 126: Calculus II, Exam II Solutions

1 a) Using integration by parts with $u = x$, $dv = e^{2x} dx$, $du = dx$, $v = (1/2)e^{2x}$, gives
$$\int x e^{2x} dx = \frac{x}{2} e^{2x} - \frac{1}{2} \int e^{2x} dx = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + C$$

b) Using the trig substitution $x = \tan(\theta)$, $dx = \sec^2(\theta)d\theta$, $\sqrt{x^2 + 1} = \sec(\theta)$, gives
$$\int \frac{\sqrt{x^2 + 1}}{x^4} dx = \int \frac{\sec^3(\theta)}{\tan^4(\theta)} d\theta = \int \frac{\cos(\theta)}{\sin^4(\theta)} d\theta.$$
 Substituting $u = \sin(\theta)$, $du = \cos(\theta)d\theta$ gives
$$\int u^{-4} du = -\frac{1}{3 \sin^3(\theta)} + C = -\frac{(x^2 + 1)^{3/2}}{3x^3} + C.$$

2. The partial fraction decomposition of $\frac{x+2}{x^2(x^2+1)}$ looks like $\frac{x+2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$. Multiplying by $x^2(x^2+1)$ gives $x+2 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 = (A+C)x^3 + (B+D)x^2 + Ax + B$. Therefore, $A = 1$, $B = 2$, $A + C = 0$, and $B + D = 0$, so $C = -1$, $D = -2$.

3.
$$\sum_{n=1}^{\infty} \frac{3^n + 2(-1)^{n+1}}{4^n} = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n - 2 \sum_{n=1}^{\infty} \left(\frac{-1}{4}\right)^n = \frac{3/4}{1 - 3/4} - 2 \frac{-1/4}{1 - (-1/4)} = 3 + 2\frac{1}{5} = \frac{17}{5}$$

4. a) $a_n = \frac{(-1)^n n}{n+1}$ diverges because one subsequence converges to 1, $a_{2k} = \frac{2k}{2k+1} \rightarrow 1$, and another subsequence converges to -1 , $a_{2k+1} = \frac{-(2k+1)}{2k+2} \rightarrow -1$.

b) $a_n = n \sin(1/n) = \frac{\sin(1/n)}{1/n}$, so applying L'Hopital's Rule gives
$$\lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{n \rightarrow \infty} \frac{\cos(1/n)(-1/n^2)}{-1/n^2} = \cos(0) = 1.$$

5. Consider $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$.

a) The 4th partial sum is $s_4 = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \ln\left(\frac{4}{5}\right) = \ln(1) - \ln(2) + \ln(2) - \ln(3) + \ln(3) - \ln(4) + \ln(4) - \ln(5) = -\ln(5)$

b) The series is telescoping, since $\ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$, so the n^{th} partial sum is $s_n = \ln(1) - \ln(n+1)$ (as in part a). Therefore, the limit of s_n is $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} -\ln(n) = -\infty$ and the series diverges.

6. a) $\sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)^n$ diverges because the n^{th} term does not approach 0:

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = e^{-1} \neq 0.$$

b) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2 - 1}$ converges by the Limit Comparison Test: We compare the series to $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ which converges because it is a p -series with $p = 3/2 > 1$. Taking the limit of the quotient of the terms of these series gives $\lim_{n \rightarrow \infty} \frac{\sqrt{n}/(n^2 - 1)}{1/n^{3/2}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 - 1} = \lim_{n \rightarrow \infty} \frac{1}{1 - 1/n^2} = 1$.

c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^3}$ converges by the Integral Test: Using the substitution $u = \ln(x)$, $du = \frac{1}{x} dx$, gives $\int_2^{\infty} \frac{1}{x(\ln(x))^3} dx = \int_{\ln(2)}^{\infty} \frac{1}{u^3} du = \lim_{b \rightarrow \infty} -\frac{1}{2u^2} \Big|_{\ln(2)}^b = \lim_{b \rightarrow \infty} -\frac{1}{2b^2} + \frac{1}{2(\ln(2))^2} = \frac{1}{2(\ln(2))^2} < \infty$.

d) $\sum_{n=1}^{\infty} \frac{5^n}{n^3}$ diverges by the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{5^{n+1}}{(n+1)^3} \frac{n^3}{5^n} = \lim_{n \rightarrow \infty} 5 \left(\frac{n}{n+1}\right)^3 = 5 > 1.$$

7. a) The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{(2n-1)!}$ converges because it is alternating, $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$, with

u_n positive and decreasing to 0: $\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(2(n+1)-1)!} \frac{(2n-1)!}{n!} = \frac{n+1}{(2n+1)(2n)} < 1$ and

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{n!}{(2n-1)!} = \lim_{n \rightarrow \infty} \frac{1}{(2n-1)(2n-2) \cdots (n+1)} = 0.$$

b) Since the error of approximating an alternating series by the n^{th} partial sum s_n is less than u_{n+1} , we can guarantee an error less than 10^{-2} by choosing n such that $u_{n+1} < 10^{-2}$.

Note that $n = 2$ is not good enough: $u_3 = \frac{3!}{5!} = \frac{1}{20} = .05 > 10^{-2}$; but $n = 3$ works: $u_4 = \frac{4!}{7!} = \frac{1}{7 \cdot 6 \cdot 5} = \frac{1}{210} = .00476 < 10^{-2}$. Therefore, $s_3 = 1 - \frac{2!}{3!} + \frac{3!}{5!} = 1 - \frac{1}{3} + \frac{1}{20} = \frac{43}{60} = .716$ approximates the series with an error less than 10^{-2} .