

Some answers for Exam II.

11. Solve the equation

$$t \frac{dy}{dt} + 2y = 5t^3$$

with initial condition $y(1) = 0$.

This is a problem from exam 1 since it is a first order linear differential equation. First put it in standard form

$$\frac{dy}{dt} + \frac{2}{t}y = 5t^2$$

so $P(t) = \frac{2}{t}$ and $Q(t) = 5t^2$.

Since $\int \frac{2}{t} dt = 2 \ln t + C$, we may take $v(t) = e^{2 \ln t} = e^{\ln t^2} = t^2$. Next we have to find $\int v(t)Q(t) dt = \int 5t^5 dt = \frac{t^6}{6} + C$. Since $y(t) = \frac{1}{v(t)} \cdot \int v(t)Q(t) dt$,

$$y(t) = \frac{t^4}{6} + \frac{C}{t^2}.$$

12. Find just the partial fraction decomposition (not the integral) of

$$\frac{2x^2 - 4x + 1}{(x - 1)^2(x - 2)}$$

First we need to set up the correct equation:

$$\frac{2x^2 - 4x + 1}{(x - 1)^2(x - 2)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x - 2}$$

Clearing denominators we have the more manageable equation

$$2x^2 - 4x + 1 = A \cdot (x - 1)(x - 2) + B \cdot (x - 2) + C \cdot (x - 1)^2$$

There are two basic strategies to pursue here: plug in judiciously chosen values to get easy equations involving A , B and C ; or equate coefficients to get three linear equations in three unknowns and solve them. Either works here but the easiest is plugging in values.

Plug in $x = 1$: $2 - 4 + 1 = B(-1)$ so $B = 1$.

Plug in $x = 2$: $8 - 8 + 1 = C$ so $C = 1$.

Plug in 0 and use the fact that $B = C = 1$: $1 = A(2) + B(-2) + C$ or $1 = 2A - 1$, so $A = 1$.

Equating coefficients goes as follows:

$$x^2: 2 = A + C.$$

$$x^1: -4 = -3A + B - 2C$$

$$x^0: 1 = 2A - 2B + C$$

There are many ways to proceed: here is one. Eliminate C using the first equation: $C = 2 - A$ so

$$-4 = -3A + B - 2(2 - A) \text{ or } 0 = -A + B.$$

$$1 = 2A - 2B + (2 - A) \text{ or } -1 = A - 2B.$$

Use the last equation to eliminate A : $A = 2B - 1$ so $0 = -(2B - 1) + B$ or $-1 = -B$, $B = 1$ and hence $A = 2B - 1 = 1$ and $C = 2 - A = 1$.

13. Using substitution, and then a trig substitution, find $\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$

Substitute to get rid of the $\ln y$: $u = \ln y$ so $du = \frac{dy}{y}$ and

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^1 \frac{du}{\sqrt{1+u^2}}$$

Now do the trig. substitution $u = \tan \theta$ so that $1 + u^2 = 1 + \tan^2 \theta = \sec^2 \theta$. The arctan takes values in the interval $(-\pi/2, \pi/2)$ and in this interval the cos, and hence the sec, are positive so $\sqrt{1+u^2} = \sec \theta$. When $u = 0$, $\theta = 0$ and when $u = 1$, $\theta = \pi/4$. Finally, $du = \sec^2 \theta d\theta$ so

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^1 \frac{du}{\sqrt{1+u^2}} = \int_0^{\pi/4} \frac{\sec^2 \theta d\theta}{\sec \theta} = \int_0^{\pi/4} \sec \theta d\theta$$

But $\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C$ so

$$\begin{aligned} \int_0^{\pi/4} \sec \theta d\theta &= \ln |\sec \theta + \tan \theta| \Big|_0^{\pi/4} \\ &= \ln(\sec \pi/4 + \tan \pi/4) - \ln(\sec 0 + \tan 0) \\ &= \ln(\sqrt{2} + 1) - \ln(1 + 0) \\ &= \ln(\sqrt{2} + 1) \end{aligned}$$

14. Does $\int_1^\infty \frac{(1 + \sin x)}{x^{4/3}} dx$ converge or diverge? Why?

This type of problem is not on exam 2 but you ought to be able to do it by now. It is a first comparison test problem so first we have to *guess* whether it converges or diverges. Since it looks roughly like $\frac{1}{x^{4/3}}$ and we know this improper integral converges, we guess that our integral converges. There is no need to justify your guess at this stage: the justification comes when you demonstrate convergence.

Since we are trying for convergence, we need a function larger than the one we have. Since $\sin x$ is between -1 and 1 ,

$$\frac{(1 + \sin x)}{x^{4/3}} \leq \frac{2}{x^{4/3}}$$

To prove our integral converges, it suffices to prove that

$$\int_1^\infty \frac{2}{x^{4/3}} dx$$

converges.

$$\begin{aligned}\int_1^\infty \frac{2}{x^{4/3}} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{2}{x^{4/3}} dx = \lim_{t \rightarrow \infty} \frac{t^{-4/3+1}}{-4/3+1} - \frac{1^{-4/3+1}}{-4/3+1} \\ &= \lim_{t \rightarrow \infty} \frac{t^{-1/3}}{-1/3} - \frac{1^{-1/3}}{-1/3} = \lim_{t \rightarrow \infty} -3t^{-1/3} + 3 = 4\end{aligned}$$

since $\lim_{t \rightarrow \infty} t^{-1/3} = 0$. Since $\int_1^\infty \frac{2}{x^{4/3}} dx$ converges, and since $\frac{(1+\sin x)}{x^{4/3}} \leq \frac{2}{x^{4/3}}$ on the interval $[1, \infty)$, the first comparison test shows $\int_1^\infty \frac{(1+\sin x)}{x^{4/3}} dx$ converges.

15. Determine the convergence of the sequence $\{a_n\}$ with $a_n = \frac{1}{n^2} \int_1^n x dx$.

This will be taken up in Chapter 8.