**11.**

a.

- The series is clearly alternating: $a_n = (-1)^n u_n$ with $u_n = \frac{1}{n \ln n}$ and $u_n > 0$ for $n \geq 2$.
- The function $\frac{1}{x \ln x}$ is decreasing since $f'(x) = \dfrac{-(\ln x + x \cdot \frac{1}{x})}{(x \ln x)^2} = \dfrac{-1 - \ln x}{(x \ln x)^2} < 0$.
- $\displaystyle\lim_{x \to \infty} \frac{1}{x \ln x} = 0$

We have checked the three hypothesis for the Alternating Series Test to hold and so the series converges.

b. To determine whether or not the series absolutely converges, we must check whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges or not. We wish to apply the Integral Test to the function $f(x) = \frac{1}{x \ln x}$. We checked in part a. that $f(x)$ is decreasing so the hypotheses of this test are satisfied. Hence the series and the integral $\displaystyle\int_2^{\infty} \frac{dx}{x \ln x}$ converge or diverge together.

The indefinite integral $\displaystyle\int \frac{dx}{x \ln x}$ can be done via the substitution $u = \ln x$: $du = \frac{dx}{x}$, so $\displaystyle\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln u + C = \ln(\ln x) + C$. Hence $\displaystyle\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{t \to \infty} \ln(\ln t) - \ln(\ln 2)$. As $t \to \infty$, $\ln t \to \infty$ and hence $\ln(\ln t) \to \infty$. Hence the integral diverges and therefore so does the series.

**12.** To evaluate the terms in the MacLaurin series, we need to evaluate the higher derivatives of $y$ at $x = 0$. By decree, $y(0) = 1$. From the differential equation $y'(0) - 2y(0) = 0$ (since it holds for all $x$) and hence $y'(0) = 2$. Differentiating the differential equation implicitly, we see $y^{(2)} - 2y' = 0$ so at $x = 0$, $y^{(2)}(0) - 2y'(0) = 0$, or $y^{(2)}(0) = 4$.

Differentiating the equation $y^{(2)} - 2y' = 0$ implicitly we see $y^{(3)} - 2y^{(2)} = 0$, so at $x = 0$, $y^{(3)}(0) - 2y^{(2)}(0) = 0$, or $y^{(3)}(0) = 8$.

Hence the first four non–zero terms in the MacLaurin series are

$$1 + \frac{2}{1!}x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

**13.** First we calculate the radius of convergence via the $n$th root (just to be different – calculating $\left|\frac{a_{n+1}}{a_n}\right|$ also works).

$$\lim_{n\to\infty} \sqrt[n]{\frac{|x-3|^{2n}}{2^n}} = \lim_{n\to\infty} \frac{|x-3|^2}{2} = \frac{|x-3|^2}{2} \ .$$

Solve $\frac{|x-3|^2}{2} = 1$, or $|x-3|^2 = 2$ so $|x-3| = \sqrt{2}$ and the radius of convergence is $\sqrt{2}$. The two endpoints of our interval are $3 \pm \sqrt{2}$ and we need to check for convergence at the end points: i.e. we need to check if the two series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(3 \pm \sqrt{2} - 3)^{2n}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(\pm\sqrt{2})^{2n}}{2^n}$ converge or not. But both the series are the same: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n$ and this series diverges. Hence the *interval* of convergence is $(3-\sqrt{2}, 3+\sqrt{2})$.

**14.**

a. We know that the MacLaurin series for $\displaystyle\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ with radius of convergence 1. Hence $\displaystyle\frac{1}{1+(-2x)} = \sum_{n=0}^{\infty} (-1)^n(-2x)^n = \sum_{n=0}^{\infty} 2^n x^n$ and this representation is valid for $|x| < 1/2$. In particular, this must be the MacLaurin series for $\frac{1}{1-2x}$.

b. Since $\displaystyle\frac{d}{dx}\frac{1}{1-2x} = \frac{2}{(1-2x)^2}$ we know that the MacLaurin series for $= \frac{2}{(1-2x)^2}$ is the derivative of the series $\displaystyle\sum_{n=0}^{\infty} 2^n x^n$ which we know is $\displaystyle\sum_{n=1}^{\infty} n \cdot 2^n x^{n-1} = \sum_{n=0}^{\infty} (n+1)2^{n+1} x^n$. Hence $\displaystyle\sum_{n=0}^{\infty} (n+1)2^{n+1} x^n$ is the MacLaurin series for $\frac{2}{(1-2x)^2}$.

c. Since $\displaystyle\frac{2x}{(1-2x)^2} = x \cdot \frac{2}{(1-2x)^2}$ the MacLaurin series for

$$\frac{2x}{(1-2x)^2} = x \sum_{n=0}^{\infty} (n+1)2^{n+1} x^n = \sum_{n=0}^{\infty} (n+1)2^{n+1} x^{n+1} = \sum_{n=1}^{\infty} n \cdot 2^n \, x^n$$

**15.** For any $x$ in the interval $[0,1]$ $\sum_{n=1}^{\infty}(-1)^n \frac{nx^{n-1}}{11^n}$ is an alternating series with $u_n = \frac{nx^{n-1}}{11^n} \geq 0$. The MacLaurin series for $F(x) = \sum_{n=1}^{\infty}(-1)^n \frac{nx^n}{n \cdot 11^n} = \sum_{n=1}^{\infty}(-1)^n \frac{x^n}{11^n}$. Hence $F(x)$ is a geometric series, but this is not particularly relevant to our current question.

More to the point, for $0 < x \leq 1$, the series is alternating with $u_n = \frac{x^n}{11^n}$. The $u_n$ decrease since $\frac{du_n}{dn} = \left(\ln\left(\frac{x}{11}\right)\right) \cdot \left(\frac{x}{11}\right)^n$. Since $0 < x < 11$, $\ln\left(\frac{x}{11}\right) < 0$ and so is $\frac{du_n}{dn}$.

Hence, for any $0 < x \leq 1$, the Alternating Series Test AND its accuracy result applies. For $x = 0$ $F(0) = 0$ and all the MacLaurin polynomial approximations are also 0 so we get infinite accuracy with any approximation. Otherwise, we know that the accuracy of the approximation is at worst the first term neglected, which in our case is $\left(\frac{x}{11}\right)^4$. This quantity is increasing as a function of $x$ so for $x$ in $[0,1]$ the error is at most $\frac{1}{11^4}$.