Math. 126 Quiz #1

January 23, 2001

Consider the region below the curve $y = \cos x$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and above the x–axis. Assume the density is constant, δ.

a) Find the mass of this region.

b) Write a definite integral whose value will be the moment about the y–axis of this region. Give a short reason why the moment about the y–axis of this region is 0 even though you do not yet know how to do the integral.

c) Write a definite integral whose value will be the moment about the x–axis of this region. You should realize that we have talked about evaluating this integral, but don't do it today.

d) Write a definite integral whose value will be the volume obtained by rotating this region about the x–axis. Use the disk method from last semester.

Remark for after the quiz. The proportionality relation between the moment about the x–axis and the volume of the solid obtained by rotation about the x–axis is true in general and goes back to Pappus of Alexandria around 300AD.

Solution

(a) $Mass = \delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \; dx = \delta(-\sin x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \delta\big(1 - (-1)\big) = 2\delta$

(b) $Moment_y = \delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x \; dx = 0$ since the region is symmetric about the y–axis.

(c) $Moment_x = \delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} \cos^2 x \; dx$

The integral can be evaluated as follows:$\delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}\big(1+\cos(2x)\big) = \frac{\delta}{4} x - \frac{1}{2} \sin 2x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi\delta}{4}$

(d) $Volume = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \; dx$

Math. 126 Quiz #2

January 30, 2001

It is true that for positive integers 2, 3, 4, ... the following holds.

$$(*) \qquad 1/2 + 1/3 + \cdots + 1/n < \ln n < 1 + 1/2 + 1/3 + \cdots + 1/n$$

1. Explain why.
2. Given that $3^5 = 243$ and that $5.0 < 1/2 + 1/3 + \cdots + 1/243$ use $(*)$ with $n = 3^5$ and the laws of logarithms to argue $e < 3$.

Solution

1. By definition $\ln n = \displaystyle\int_1^n \frac{dx}{x}$. Recall $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the right–hand Riemann sum for the partition of $[1, n]$ into n pieces of length 1: $f'(x) = \frac{-1}{x^2} < 0$. $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$ is the left–hand Riemann sum for the partition of $[1, n]$ into n pieces of length 1. The graph of $f(x) = \frac{1}{x}$ is decreasing since $f'(x) = \frac{-1}{x^2} < 0$, so the right–hand Riemann sum is less than the integral and the left hand Riemann sum is greater. Hence $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

2. Using the facts given, we see $5.0 < \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{3^5} < \ln 3^5$. Hence $5.0 < \ln 3^5 = 5 \cdot \ln 3$ and therefore $1 < \ln 3$. Since $\ln e = 1$ we see $\ln e < \ln 3$ and since ln is an increasing function ($\frac{d \ln x}{dx} = \frac{1}{x} > 0$) $e < 3$.

Math. 126 Quiz #3

February 6, 2001

1. Solve $6^{5k} = 4$. Leave your answer as a quotient involving numbers and natural logs of numbers.
2. Compute $\frac{dy}{dx}$ where $y = x^{e^x}$.

Solution

1. $\ln 6^{5k} = \ln 4$ so $5k \ln 6 = \ln 4$ or $k = \dfrac{\ln 4}{5 \ln 6}$
2. Rewrite $y = x^{e^x} = e^{e^x \ln x}$ so $\dfrac{dy}{dx} = \left(e^{e^x \ln x}\right) \dfrac{d\, e^x \ln x}{dx}$.

$\frac{d\, e^x \ln x}{dx} = e^x \ln x + e^x \frac{1}{x}$, so

$$\frac{dy}{dx} = x^{e^x} \left(e^x \ln x + \frac{e^x}{x}\right)$$

Math. 126 Quiz #4

February 13, 2001

Solve the initial value problem

$$x \frac{dy}{dx} = x^2 \cos x - y$$

$$y(\pi) = 1$$

Solution

First put it in standard form: $\dfrac{dy}{dx} + \dfrac{y}{x} = x \cos x$

Then $P = \dfrac{1}{x}$ and $Q = x \cos x$. Compute $\int P dx = \ln x + C$ so we may take $v = e^{\ln x} = x$.

Then $\int vQ dx = \int \cos x \, dx = \sin x + C$.

Hence $y = \dfrac{1}{v} \int vQ dx = \dfrac{\sin x}{x} + \dfrac{C}{x}$ so $y(\pi) = \dfrac{\sin \pi}{\pi} + \dfrac{C}{\pi} = \dfrac{C}{\pi} = 1$. Hence $C = \pi$ and

$$y = \frac{\sin x}{x} + \frac{\pi}{x}$$

Math. 126 Quiz #5
February 27, 2001

Evaluate the integral

$$\int \sin(\sqrt{x}\,)\;dx\ .$$

Hint: First do a substitution and then an integration by parts.

Solution

Substitute $w = \sqrt{x}$: then $dw = \dfrac{dx}{2\sqrt{x}}$, so $2\sqrt{x}\;dw = dx$ and $2wdw = dx$. Hence

$$\int \sin(\sqrt{x}\,)\;dx = 2\int w\sin(w)\;dw$$

Parts: $\begin{array}{cc} u = w & du = dw \\ dv = \sin(w)dw & v = -\cos(w) \end{array}$ so

$$\begin{aligned} 2\int w\sin(w)\;dw &= -\,2w\cos w + 2\int \cos w\;dw \\ &= -\,2w\cos w + 2\sin w + C \end{aligned}$$

Finally

$$\int \sin(\sqrt{x}\,)\;dx = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C\ .$$

Math. 126 Quiz #6

March 6, 2001

Expand

$$\frac{2x^3 - 2x^2 + 3x + 1}{(x^2 - x - 2)(x^2 + 1)}$$

as a sum of partial fractions.

Solution

$$\frac{2x^3 - 2x^2 + 3x + 1}{(x^2 - x - 2)(x^2 + 1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)} + \frac{Cx + D}{(x^2+1)}$$

$$2x^3 - 2x^2 + 3x + 1 = A(x+1)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x+1)(x-2)$$

Equate coefficients:

$$\begin{aligned} x^3 &: 2 = A + B + C \\ x^2 &: -2 = A - 2B + D - C \\ x^1 &: 3 = A + B - 2C - D \\ x^0 &: 1 = A - 2B - 2D \end{aligned}$$

From x^3: $A = 2 - B - C$ so the other equations become

$$\begin{aligned} x^2 : \quad & -2 = 2 - B - C - 2B + D - C \\ & -4 = -3B - 2C + D \\ x^1 : \quad & 3 = 2 - B - C + B - 2C - D \\ & 1 = -3C - D \\ x^0 : \quad & 1 = 2 - B - C - 2B - 2D \\ & -1 = -3B - 2D \end{aligned}$$

From x^1: $D = -3C - 1$ so

$$\begin{aligned} x^2 : \quad & -4 = -3B - 2C + -3C - 1 \\ & -3 = -3B - 5C \\ x^0 : \quad & -1 = -3B - 2(-3C - 1) \\ & -3 = -3B + 6C \end{aligned}$$

It follows that $C = 0$ and $B = 1$, whence $D = -1$ and $A = 1$.

Plug in:

$x = 2 : 16 - 8 + 6 + 1 = A(3)(5)$ or $15 = 15A$ or $A = 1$.

$x = -1 : -2 - 2 - 3 + 1 = B(-3)(2)$ or $-6 = (-6)B$ or $B = 1$.

$x = 0 : 1 = A + (-2)B + (-2)D$ or $1 = -1 + (-2)D$ or $D = -1$.

$x = 1 : 2 - 2 + 3 + 1 = A(2)(2) + B(-1)(2) + (C + D)(2)(-1)$ or $4 = 4 - 2 - 2(C - 1)$ or $2 = 2 - 2C$ or $C = 0$.

Math 126, Quiz #7
March 27, 2001

Which improper integrals below converge and which diverge? A brief indication of your reasoning should be given.

a) $\int_0^\infty e^{-x^3}\,dx$

b) $\int_0^\infty \frac{1}{\sqrt[3]{x^2+1}}\,dx$

Solution

a) $\int_0^\infty e^{-x^3}\,dx$ converges if and only if $\int_1^\infty e^{-x^3}\,dx$ converges.

On the interval $[1,\infty)$, $x \geq 1$ so $x^2 \geq 1$ and $x^3 \geq x$ so $e^x \leq e^{x^3}$ so $e^{-x^3} \leq e^{-x}$. Now $\int_1^\infty e^{-x}\,dx$ converges since $\lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty} -e^{-x}\Big|_1^t = e^{-1} - \lim_{t\to\infty} e^{-t} = e^{-1} - 0$. By the first comparison test for improper integrals, $\int_1^\infty e^{-x^3}\,dx$ converges and hence so does $\int_0^\infty e^{-x^3}\,dx$.

b) Roughly speaking $\frac{1}{\sqrt[3]{x^2+1}}$ behaves near ∞ like $x^{-2/3}$. More precisely, $\lim_{x\to\infty}\frac{\frac{1}{\sqrt[3]{x^2+1}}}{x^{-2/3}} = \lim_{x\to\infty}\frac{x^{2/3}}{\sqrt[3]{x^2+1}} = \lim_{x\to\infty}\frac{1}{\sqrt[3]{1+\frac{1}{x^2}}} = 1$. Since $0 < 1 < \infty$ we would like to use the limit comparison test, comparing the integral for $\frac{1}{\sqrt[3]{x^2+1}}$ with the one for $x^{-2/3}$. Annoyingly, $x^{-2/3}$ has an additional singularity at 0 so we proceed as follows.

$\int_0^\infty \frac{1}{\sqrt[3]{x^2+1}}\,dx$ converges if and only if $\int_1^\infty \frac{1}{\sqrt[3]{x^2+1}}\,dx$ converges and by the limit comparison test for improper integrals, $\int_1^\infty \frac{1}{\sqrt[3]{x^2+1}}\,dx$ converges if and only if $\int_1^\infty x^{-2/3}\,dx$ converges.

But $\int_1^\infty x^{-2/3}\,dx = \lim_{t\to\infty}\frac{x^{1/3}}{1/3}\Big|_1^t \lim_{t\to\infty} 3 - 3t^{1/3} = 3 - \infty$ so $\int_1^\infty x^{-2/3}\,dx$ diverges and hence so does $\int_0^\infty \frac{1}{\sqrt[3]{x^2+1}}\,dx$.

Math. 126 Quiz #8

April 3, 2001

For each of the series below, do two things. First compute $\lim_{n\to\infty} a_n$ and then use this calculation to say if you are **certain** that $\sum_{n=1}^{\infty} a_n$ diverges or if the limit calculation does not suffice to say if the series converges or diverges. Just circle **diverges** or **can not tell** after the series for the second part of each question

A. $a_n = \frac{1}{\sqrt{n}}$:

a. $\lim_{n\to\infty} a_n =$

b. $\sum_{n=1}^{\infty} a_n$ **diverges** **can not tell**

B. $a_1 = 0$ and $a_n = a_{n-1} + 1$ for all $n \geq 2$:

Solution

A. $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \sqrt{\lim_{n\to\infty} \frac{1}{n}} = \sqrt{0} = 0$. Since the terms go to 0, we can not tell if the series diverges or converges. We know from our later work that this is a p–series with $p = \frac{1}{2} \leq 1$ so it does diverge, but not because of this calculation.

B. Since $a_n > a_{n-1}$ the series is increasing so either $\lim_{n\to\infty} a_n = \infty$ or it exists. In either case, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n + 1$, which implies $\lim_{n\to\infty} a_n = \infty$. A second way to do this calcualtion is to observe that $a_n = n - 1$ and hence $\lim_{n\to\infty} a_n = \infty$. In any case $\lim_{n\to\infty} a_n \neq 0$ so we are sure that this series diverges.

Math. 126 Quiz #9
April 10, 2001

Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^2+1}}$ converges by *checking* the hypotheses of the Alternating Series Test.

Then show the calcualtions needed to find an m such that

$$0 < \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^2+1}} - \sum_{n=1}^{m} \frac{(-1)^{n+1}}{\sqrt{n^2+1}} < 0.01$$

Solution

For the Alternating Series Test, $a_n = \frac{(-1)^{n+1}}{\sqrt{n^2+1}} = (-1)^{n+1}u_n$ with $u_n = \frac{1}{\sqrt{n^2+1}} > 0$, so the series is alternating. The limit $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$ so the terms go to 0. Finally we check that $u_{n+1} < u_n$: let $f(x) = \frac{1}{\sqrt{x^2+1}}$ so $u_n = f(n)$. Compute $f'(x) = -\frac{3}{2}(x^2+1)^{-3/2}(2x)$ and $f'(x) > 0$ for $x > 0$. Hence f is decreasing on the interval $[0, \infty)$ so $u_{n+1} = f(n+1) > f(n) = u_n$.

Since the difference indicated is positive, m must be odd and for odd m

$$0 < \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^2+1}} - \sum_{n=1}^{m} \frac{(-1)^{n+1}}{\sqrt{n^2+1}} < u_{m+1}$$

Hence any m odd such that $u_{m+1} \leq 0.01 = \frac{1}{100}$ will suffice: $\frac{1}{\sqrt{m^2+1}} \leq \frac{1}{100}$ or $\frac{1}{m^2+1} \leq \frac{1}{10^4}$, or $10^4 \leq m^2 + 1$. The smallest integer satisfying this inequality is $m = 100$ so $m = 101$ is the smallest integer satisfying both our requirements.

Math. 126 Quiz #10
April 17, 2001

Compute the interval of convergence of each of the following power series. For each series, indicate where the convergence is conditional and where it is absolute.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{7^n}$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(n+1)7^n}$

Solution

First compute the radii of convergence. For a. we calculate $\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{x^{2n}}{7^n}} = \lim_{n\to\infty} \frac{x^2}{7}$ so $\frac{R^2}{7} = 1$ or $R = \sqrt{7}$. For b. we calculate $\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{x^{2n}}{(n+1)7^n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n+1}} \cdot \frac{x^2}{7} = \frac{x^2}{7}$ so again$R = \sqrt{7}$.

At the endpoints in a. the series to consider are $\displaystyle\sum_{n=0}^{\infty} \frac{(\pm\sqrt{7}^{2n}}{7^n} = \sum_{n=0}^{\infty} 1$ and both of these series diverge. At the endpoints in part b. the series to consider are $\displaystyle\sum_{n=0}^{\infty} \frac{(\pm\sqrt{7}^{2n}}{(n+1)7^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}$. Both of these series are the harmonic series and thus diverge. Therefore, in both cases, the interval of convergence is $(-\sqrt{7}, \sqrt{7})$.

By our theory, the convergence is absolute on the open interval (always the case for a power series) so in both cases the convergence is absolute on $(-\sqrt{7}, \sqrt{7})$ and divergent elsewhere. Neither series conditionally converges anywhere.

Math. 126 Quiz #11
May 1, 2001

The MacLaurin series for $\sin x$ is

$$\sin x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

a. What is the radius of convergence of this series?
(Just an answer is sufficient - no reason need be given.)
b. Write down the MacLaurin series for $\frac{\sin x}{x}$. Give a reason why it is the MacLaurin series.
c. Write down the MacLaurin series for the function $\displaystyle\int_0^x \frac{\sin t^2}{t^2}\, dt$.

Solution

a. The radius of convergence of the MacLaurin series for $\sin x$ is ∞. If you want to check this:

$$\lim_{n\to\infty} \frac{\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{x^{2n+1}}{(2n+1)!}} = \lim_{n\to\infty} \frac{\frac{x^{2n+3}}{(2n+3)!}}{\frac{x^{2n+1}}{(2n+1)!}} = \lim_{n\to\infty} x^2 \cdot \frac{(2n+1)!}{(2n+3)!}$$

$$= \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} = 0$$

b. $\displaystyle\frac{\sin x}{x} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$ is a valid equation since we are just dividing an equality (the MacLaurin series for $\sin x$) by x. Hence we have a power series centered at 0 for $\frac{\sin x}{x}$ and this *must* be the MacLaurin series because a power series is its own Taylor series.

Remark: To be completely precise, we should have written the function $\frac{\sin x}{x}$ as

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

We know from 125 that $f(x)$ is continuous at 0. The power series certainly represents $f(x)$ if $x \neq 0$ and equality at $x = 0$ follows by evaluating the power series at 0. Hence the power series represents $f(x)$ everywhere and is therefore its MacLaurin series. Notice that as a side benefit we have shown that $f(x)$ is infinitely differentiable (this being obvious except at $x = 0$).

c. We are actually using the function $f(x)$ from the remark but we will write it as $\frac{\sin x}{x}$. Using term–for–term integration,

$$\int_0^x \frac{\sin t^2}{t^2}\, dt = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot(2n+1)!}$$

is a valid equation and hence the series on the right is the MacLaurin series for the function on the left.