Answers to Exam I
Spring 2000

11. Consider the function $f(x) = \sqrt{2x^4 + x^2}$.
a) Show that f is one to one on the domain $(0, \infty)$.
b) Find the slope of the tangent line to the graph of the inverse function f^{-1} at the point $f^{-1}(6) = 2$.

First compute $\dfrac{d}{dx}\sqrt{2x^4 + x^2} = \dfrac{8x^3 + 2x}{2\sqrt{2x^4 + x^2}}$

(a) Since $x > 0$, $8x^3 + 2x > 0$ so f is increasing on $(0, \infty)$. Therefore f is one–to–one. More precisely, if $f(x_1) = f(x_2)$ then it is not true that $x_1 > x_2$ (since then $f(x_1) > f(x_2)$ and this does not hold). It is also not true that $x_1 < x_2$, so it follows that $x_1 = x_2$.

(b) The general formula says

$$\frac{d}{dx} f^{-1}\big(f(x)\big) = \frac{1}{f'(x)} \ .$$

Hence we need to find an x such that $f(x) = 6$ and we are told that $x = 2$ will work (and since f is one–to–one, this is the only x that works). Hence the answer is

$$\frac{1}{f'(2)} = \frac{1}{\frac{8x^3+2x}{2\sqrt{2x^4+x^2}}}\Bigg|_{x=2} = \frac{1}{\frac{8(2)^3+2(2)}{2\sqrt{2(2)^4+(2)^2}}} = \frac{1}{\frac{8(8)+4}{2\sqrt{2(16)+4}}} = \frac{1}{\frac{68}{2\sqrt{36}}} = \frac{1}{\frac{34}{6}} = \frac{1}{\frac{17}{3}} = \frac{3}{17}$$

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12. Find the derivative of the function

$$f(x) = \sqrt[x]{x} = x^{\frac{1}{x}} \quad .$$

This is an example of an exponential function that you do not understand very well, so rewrite it to base e:

$$x^{\frac{1}{x}} = e^{(\ln x)\frac{1}{x}} = e^{\frac{\ln x}{x}} \ .$$

Hence $\dfrac{d}{dx}x^{\frac{1}{x}} = e^{\frac{\ln x}{x}}\left(\dfrac{d}{dx}\dfrac{\ln x}{x}\right) = x^{\frac{1}{x}}\left(\dfrac{(\frac{1}{x})x - (\ln x)(1)}{x^2}\right) = x^{\frac{1}{x}-2}\big(1 - \ln x\big)$.

$$\frac{d}{dx}x^{\frac{1}{x}} = x^{\frac{1-2x}{x}}\big(1 - \ln x\big).$$

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13. The quantity of a radioactive substance decreases from 100% to 80% in three hours. Compute the half–life (the time until you have 50% of your sample left) as a quotient of logs.

Radioactive decay is an example of exponential decay so the equation governing the amount is

$$A(t) = A_0 e^{-kt} \ .$$

The fact that we have decreased to 80% in 3 hours means that $A(3) = 0.8 \cdot A_0$. But $A(3) = A_0 e^{3k}$ so $e^{3k} = 0.8$. Then $3k = \ln(0.8)$ so $k = \frac{\ln(0.8)}{3}$. To find the half–life, T_0, solve $A(T_0) = 0.5 \cdot A_0$ or $e^{kT_0} = 0.5$. Hence $kT_0 = \ln(0.5)$ so

$$T_0 = \frac{3\ln(0.5)}{\ln(0.8)}$$

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14. Determine $\lim\limits_{x\to\infty} \sqrt{e^x + x} - \sqrt{e^x + 1}$.

Hint: Rewrite the expression using algebra and then use what you know about rates of growth.

This is not material we have covered yet, but ....

$$\begin{aligned}
\sqrt{e^x + x} - \sqrt{e^x + 1} &= \sqrt{e^x + x} - \sqrt{e^x + 1} \ \frac{\sqrt{e^x + x} + \sqrt{e^x + 1}}{\sqrt{e^x + x} + \sqrt{e^x + 1}} \\
&= \frac{e^x + x - (e^x + 1)}{\sqrt{e^x + x} + \sqrt{e^x + 1}} \\
&= \frac{x - 1}{\sqrt{e^x + x} + \sqrt{e^x + 1}}
\end{aligned}$$

and we knew from work at this point last year that exponentials grow more quickly than polynomials, so the limit is 0.

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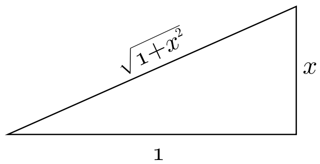
15. Express $\sec\bigl(\arctan(x)\bigr)$ as an algebraic function of x.

Another problem of a sort to which we did not get for this exam, but ...
To solve this problem, we first construct a triangle



From this triangle we see that $\sec\bigl(\arctan(x)\bigr) = \frac{\pm\sqrt{1+x^2}}{1} = \pm\sqrt{1+x^2}$. The arctan takes on values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and for these angles, the cos is positive, hence so is the sec and we get

$$\sec\bigl(\arctan(x)\bigr) = \sqrt{1+x^2}\ .$$