Some answers for Exam II.

11. Solve the equation

$$t\frac{dy}{dt} + 2y = 5t^3$$

with initial condition $y(1) = 0$.

This is a problem from exam 1 since it is a first order linear differential equation. First put it in standard form

$$\frac{dy}{dt} + \frac{2}{t}y = 5t^2$$

so $P(t) = \frac{2}{t}$ and $Q(t) = 5t^2$.

Since $\displaystyle\int \frac{2}{t}\,dt = 2\ln t + C$, we may take $v(t) = e^{2\ln t} = e^{\ln t^2} = t^2$. Next we have to find $\displaystyle\int v(t)Q(t)\,dt = \int 5t^5\,dt = \frac{t^6}{6} + C$. Since $y(t) = \frac{1}{v(t)} \cdot \int v(t)Q(t)\,dt$,

$$y(t) = \frac{t^4}{6} + \frac{C}{t^2}\ .$$

12. Find just the partial fraction decomposition (not the integral) of

$$\frac{2x^2 - 4x + 1}{(x-1)^2(x-2)}$$

First we need to set up the correct equation:

$$\frac{2x^2 - 4x + 1}{(x-1)^2(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)}$$

Clearing denominators we have the more manageable equation

$$2x^2 - 4x + 1 = A \cdot (x-1)(x-2) + B \cdot (x-2) + C \cdot (x-1)^2$$

There are two basic strategies to pursue here: plug in judiciously chosen values to get easy equations involving A, B and C; or equate coefficients to get three linear equations in three unknowns and solve them. Either works here but the easiest is plugging in values.

Plug in $x = 1$: $2 - 4 + 1 = B(-1)$ so $B = 1$.
Plug in $x = 2$: $8 - 8 + 1 = C$ so $C = 1$.
Plug in 0 and use the fact that $B = C = 1$: $1 = A(2) + B(-2) + C$ or $1 = 2A - 1$, so $A = 1$.

Equating coefficients goes as follows:
x^2: $2 = A + C$.
x^1: $-4 = -3A + B - 2C$
x^0: $1 = 2A - 2B + C$

There are many ways to proceed: here is one. Eliminate C using the first equation: $C = 2 - A$ so

$-4 = -3A + B - 2(2 - A)$ or $0 = -A + B$.

$1 = 2A - 2B + (2 - A)$ or $-1 = A - 2B$.

Use the last equation to eliminate A: $A = 2B - 1$ so $0 = -(2B - 1) + B$ or $-1 = -B$, $B = 1$ and hence $A = 2B - 1 = 1$ and $C = 2 - A = 1$.

13. Using substitution, and then a trig substitution, find $\displaystyle\int_1^e \frac{dy}{y\sqrt{1 + (\ln y)^2}}$

Substitute to get rid of the $\ln y$: $u = \ln y$ so $du = \frac{dy}{y}$ and

$$\int_1^e \frac{dy}{y\sqrt{1 + (\ln y)^2}} = \int_0^1 \frac{du}{\sqrt{1 + u^2}}$$

Now do the trig. substitution $u = \tan\theta$ so that $1 + u^2 = 1 + \tan^2\theta = \sec^2\theta$. The arctan takes values in the interval $(-\pi/2, \pi/2)$ and in this interval the cos, and hence the sec, are positive so $\sqrt{1 + u^2} = \sec\theta$. When $u = 0$, $\theta = 0$ and when $u = 1$, $\theta = \pi/4$. Finally, $du = \sec^2\theta\, d\theta$ so

$$\int_1^e \frac{dy}{y\sqrt{1 + (\ln y)^2}} = \int_0^1 \frac{du}{\sqrt{1 + u^2}} = \int_0^{\pi/4} \frac{\sec^2\theta d\theta}{\sec\theta} = \int_0^{\pi/4} \sec\theta\, d\theta$$

But $\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$ so

$$\begin{aligned}\int_0^{\pi/4} \sec\theta\, d\theta &= \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4} \\ &= \ln\big(\sec\pi/4 + \tan\pi/4\big) - \ln\big(\sec 0 + \tan 0\big) \\ &= \ln(\sqrt{2} + 1) - \ln(1 + 0) \\ &= \ln(\sqrt{2} + 1)\end{aligned}$$

14. Does $\displaystyle\int_1^\infty \frac{(1 + \sin x)}{x^{4/3}} dx$ converge or diverge? Why?

This type of problem is not on exam 2 but you ought to be able to do it by now. It is a first comparison test problem so first we have to *guess* whether it converges or diverges. Since it looks roughly like $\frac{1}{x^{4/3}}$ and we know this improper integral converges, we guess that our integral converges. There is no need to justify your guess at this stage: the justification comes when you demonstrate convergence.

Since we are trying for convergence, we need a function larger than the one we have. Since $\sin x$ is between -1 and 1,

$$\frac{(1 + \sin x)}{x^{4/3}} \le \frac{2}{x^{4/3}}$$

To prove our integral converges, it suffices to prove that

$$\int_1^\infty \frac{2}{x^{4/3}}\, dx$$

converges.

$$\int_1^\infty \frac{2}{x^{4/3}}\,dx = \lim_{t\to\infty}\int_1^t \frac{2}{x^{4/3}}\,dx = \lim_{t\to\infty}\frac{t^{-4/3+1}}{-4/3+1} - \frac{1^{-4/3+1}}{-4/3+1}$$
$$= \lim_{t\to\infty}\frac{t^{-1/3}}{-1/3} - \frac{1^{-1/3}}{-1/3} = \lim_{t\to\infty} -3t^{-1/3} + 3 = 4$$

since $\lim\limits_{t\to\infty} t^{-1/3} = 0$. Since $\displaystyle\int_1^\infty \frac{2}{x^{4/3}}dx$ converges, and since $\frac{(1+\sin x)}{x^{4/3}} \leq \frac{2}{x^{4/3}}$ on the interval $[1,\infty)$, the first comparison test shows $\displaystyle\int_1^\infty \frac{(1+\sin x)}{x^{4/3}}dx$ converges.

15. Determine the convergence of the sequence $\{a_n\}$ with $a_n = \dfrac{1}{n^2}\displaystyle\int_1^n x\ dx$.

This will be taken up in Chapter 8.