Math 126
Exam II
March 20, 2001

9. First by long division the integrand is equal to

$$2x + \frac{1}{x(x-1)}.$$

Expand the second term using partial fractions:

$$\frac{1}{x(x-1)} = \frac{A}{x-1} + \frac{B}{x}$$

or

$$1 = Ax + B(x-1)\ .$$

Solving the equations gives

$$\frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x}.$$

We can then use standard integration formulas to get the general antiderivative

$$x^2 + \ln|x-1| - \ln|x| + C.$$

---

10. Use the substitution $x = \frac{3}{4}\sin\theta$: then $dx = \frac{3}{4}\cos\theta\, d\theta$ and

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-16x^2}} &= \left(\frac{3}{4}\right)^2 \int \frac{\sin^2\theta\ \cos\theta\ d\theta}{\sqrt{9-9\sin^2\theta}} \\
&= \frac{9}{64}\int \sin^2\theta\ d\theta \\
&= \frac{9}{128}\big(\theta - \sin\theta\ \cos\theta\big) + C \quad \text{(from memory or see below)} \\
&= \frac{9}{128}\left(\arcsin\frac{4x}{3} - \frac{4x}{3}\sqrt{1-\left(\frac{4x}{3}\right)^2}\ \right) + C \\
&= \frac{9}{128}\left(\arcsin\frac{4x}{3} - \frac{4x}{9}\sqrt{9-16x^2}\ \right) + C
\end{aligned}$$

The integral $\int \sin^2\theta\, d\theta$ can be computed several different ways if you didn't memorize the answer.

1. Use the trig. identities $\sin^2\theta = \frac{1}{2}\big(1-\cos(2\theta)\big)$ and $\sin(2\theta) = 2\sin\theta\cos\theta$: $\displaystyle\int \sin^2\theta\, d\theta = \int \frac{1}{2}\big(1-\cos(2\theta)\big)\ d\theta = \frac{1}{2}\Big(\theta - \frac{1}{2}\sin(2\theta)\Big) + C = \frac{1}{2}\big(\theta - \sin\theta\ \cos\theta\big) + C$
2. Use integration by parts: $u = \sin\theta$, $dv = \sin\theta\, d\theta$; $\displaystyle\int \sin^2\theta\, d\theta = -\sin\theta\cos\theta + \int \cos^2\theta\, d\theta = -\sin\theta\cos\theta + \int 1 - \sin^2\theta\, d\theta = -\sin\theta\cos\theta + \theta - \int \sin^2\theta\ d\theta$ so $\displaystyle 2\int \sin^2\theta\ d\theta = \theta - \sin\theta\ \cos\theta + C$.

11.

$$\int e^{\sqrt{x}}\,dx \underset{\boxed{\begin{array}{c}\text{Substitution}\\ u=\sqrt{x}\\ du=\frac{1}{2\sqrt{x}}dx\\ 2\sqrt{x}\,du=dx\\ 2u\,du=dx\end{array}}}{=} \int 2u\,e^{u}\,du \underset{\boxed{\begin{array}{c}\text{Parts}\\ w=2u\\ dv=e^{u}\,du\\ dw=2du\\ v=e^{u}\end{array}}}{=} 2ue^{u}-\int 2e^{u}\,du = 2ue^{u}-2e^{u}+C$$

$$= \quad 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

This problem CAN be done via integration by parts, but it is so weird that no one actually did it this way. Still

$$\int e^{\sqrt{x}}\,dx = \int \underbrace{\sqrt{x}}_{u}\;\underbrace{\frac{1}{\sqrt{x}}e^{\sqrt{x}}\,dx}_{dv} \quad = 2\sqrt{x}\,e^{\sqrt{x}} - \int \frac{1}{\sqrt{x}}e^{\sqrt{x}}\,dx$$

$$du = \tfrac{1}{2\sqrt{x}}\ ;\ v = 2e^{\sqrt{x}}$$

$$= \quad 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

---

12. Since the answer is not obvious, rewrite:

$$\lim_{x\to 0}(1+2x)^{1/x} = e^{\lim\limits_{x\to 0}\ln(1+2x)^{1/x}}$$

and we concentrate on $\lim\limits_{x\to 0}\ln(1+2x)^{1/x} = \lim\limits_{x\to 0}\dfrac{\ln(1+2x)}{x}$.

Since $\ln(1+2\cdot 0) = \ln(1) = 0$ this last limit is of the form $\frac{0}{0}$ so we may apply l'Hôpital's rule. This requires us to compute

$$\lim_{x\to 0}\frac{\frac{d\ln(1+2x)}{dx}}{\frac{dx}{dx}} = \lim_{x\to 0}\frac{\frac{2}{1+2x}}{1} = 2\ .$$

Hence

$$\lim_{x\to 0}(1+2x)^{1/x} = e^2\ .$$

Another way to go, which no one completed successfully, is to *remember* $\lim\limits_{t\to\infty}\left(1+\dfrac{a}{t}\right)^t = e^a$ and proceed as follows. First compute $\lim\limits_{x\to 0^+}(1+2x)^{1/x}$ by letting $x = \frac{1}{t}$ and $\lim\limits_{x\to 0^+}(1+2x)^{1/x} = \lim\limits_{t\to\infty}\left(1+\dfrac{2}{t}\right)^t = e^2$. After this, you know $\lim\limits_{x\to 0}(1+2x)^{1/x} = e^2$ OR it does not exist depending on what happens with $\lim\limits_{x\to 0^-}(1+2x)^{1/x} = \lim\limits_{t\to -\infty}\left(1+\dfrac{2}{t}\right)^t$. If we let $T = -t$, then $\lim\limits_{t\to -\infty}\left(1+\dfrac{2}{t}\right)^t = \lim\limits_{T\to\infty}\left(1+\dfrac{-2}{T}\right)^{-T} = \dfrac{1}{\lim\limits_{T\to\infty}\left(1+\dfrac{-2}{T}\right)^T} = \dfrac{1}{e^{-2}} = e^2$ so $\lim\limits_{x\to 0^-}(1+2x)^{1/x} = e^2$ and $\lim\limits_{x\to 0}(1+2x)^{1/x} = e^2$.
