Math. 126 Quiz #1
January 23, 2001

Consider the region below the curve $y = \cos x$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and above the x–axis. Assume the density is constant, δ.

a) Find the mass of this region.
b) Write a definite integral whose value will be the moment about the y–axis of this region. Give a short reason why the moment about the y–axis of this region is 0 even though you do not yet know how to do the integral.
c) Write a definite integral whose value will be the moment about the x–axis of this region. You should realize that we have talked about evaluating this integral, but don't do it today.
d) Write a definite integral whose value will be the volume obtained by rotating this region about the x–axis. Use the disk method from last semester.

Remark for after the quiz. The proportionality relation between the moment about the x–axis and the volume of the solid obtained by rotation about the x–axis is true in general and goes back to Pappus of Alexandria around 300AD.

Solution

(a) $Mass = \delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \; dx = \delta(-\sin x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \delta\big(1 - (-1)\big) = 2\delta$

(b) $Moment_y = \delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x \; dx = 0$ since the region is symmetric about the y–axis.

(c) $Moment_x = \delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} \cos^2 x \; dx$

The integral can be evaluated as follows: $\delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}\big(1+\cos(2x)\big) = \frac{\delta}{4} x - \frac{1}{2} \sin 2x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi\delta}{4}$

(d) $Volume = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \; dx$

Math. 126 Quiz #2
January 30, 2001

It is true that for positive integers 2, 3, 4, ... the following holds.

$$(*) \qquad 1/2 + 1/3 + \cdots + 1/n < \ln n < 1 + 1/2 + 1/3 + \cdots + 1/n$$

1. Explain why.
2. Given that $3^5 = 243$ and that $5.0 < 1/2 + 1/3 + \cdots + 1/243$ use $(*)$ with $n = 3^5$ and the laws of logarithms to argue $e < 3$.

Solution

1. By definition $\ln n = \displaystyle\int_1^n \frac{dx}{x}$. Recall $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the right–hand Riemann sum for the partition of $[1, n]$ into n pieces of length 1: $f'(x) = \frac{-1}{x^2} < 0$. $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$ is the left–hand Riemann sum for the partition of $[1, n]$ into n pieces of length 1. The graph of $f(x) = \frac{1}{x}$ is decreasing since $f'(x) = \frac{-1}{x^2} < 0$, so the right–hand Riemann sum is less than the integral and the left hand Riemann sum is greater. Hence $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

2. Using the facts given, we see $5.0 < \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{3^5} < \ln 3^5$. Hence $5.0 < \ln 3^5 = 5 \cdot \ln 3$ and therefore $1 < \ln 3$. Since $\ln e = 1$ we see $\ln e < \ln 3$ and since ln is an increasing function ($\frac{d \ln x}{dx} = \frac{1}{x} > 0$) $e < 3$.

Math. 126 Quiz #3

February 6, 2001

1. Solve $6^{5k} = 4$. Leave your answer as a quotient involving numbers and natural logs of numbers.
2. Compute $\frac{dy}{dx}$ where $y = x^{e^x}$.

Solution

1. $\ln 6^{5k} = \ln 4$ so $5k \ln 6 = \ln 4$ or $k = \dfrac{\ln 4}{5 \ln 6}$
2. Rewrite $y = x^{e^x} = e^{e^x \ln x}$ so $\dfrac{dy}{dx} = \left(e^{e^x \ln x}\right) \dfrac{d\, e^x \ln x}{dx}$.

$\frac{d\, e^x \ln x}{dx} = e^x \ln x + e^x \frac{1}{x}$, so

$$\frac{dy}{dx} = x^{e^x} \left(e^x \ln x + \frac{e^x}{x}\right)$$

Math. 126 Quiz #4

February 13, 2001

Solve the initial value problem

$$x \frac{dy}{dx} = x^2 \cos x - y$$

$$y(\pi) = 1$$

Solution

First put it in standard form: $\dfrac{dy}{dx} + \dfrac{y}{x} = x \cos x$

Then $P = \dfrac{1}{x}$ and $Q = x \cos x$. Compute $\displaystyle\int P dx = \ln x + C$ so we may take $v = e^{\ln x} = x$.

Then $\displaystyle\int vQ dx = \int \cos x \, dx = \sin x + C$.

Hence $y = \dfrac{1}{v} \displaystyle\int vQ dx = \dfrac{\sin x}{x} + \dfrac{C}{x}$ so $y(\pi) = \dfrac{\sin \pi}{\pi} + \dfrac{C}{\pi} = \dfrac{C}{\pi} = 1$. Hence $C = \pi$ and

$$y = \frac{\sin x}{x} + \frac{\pi}{x}$$

Math. 126 Quiz #5

February 27, 2001

Evaluate the integral

$$\int \sin(\sqrt{x}\,)\;dx\;.$$

Hint: First do a substitution and then an integration by parts.

Solution

Substitute $w = \sqrt{x}$: then $dw = \dfrac{dx}{2\sqrt{x}}$, so $2\sqrt{x}\;dw = dx$ and $2wdw = dx$. Hence

$$\int \sin(\sqrt{x}\,)\;dx = 2\int w\sin(w)\;dw$$

Parts: $\begin{array}{cc} u = w & du = dw \\ dv = \sin(w)dw & v = -\cos(w) \end{array}$ so

$$\begin{aligned} 2\int w\sin(w)\;dw &= -\,2w\cos w + 2\int \cos w\;dw \\ &= -\,2w\cos w + 2\sin w + C \end{aligned}$$

Finally

$$\int \sin(\sqrt{x}\,)\;dx = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C\;.$$

Math. 126 Quiz #6

March 6, 2001

Expand

$$\frac{2x^3 - 2x^2 + 3x + 1}{(x^2 - x - 2)(x^2 + 1)}$$

as a sum of partial fractions.

Solution

$$\frac{2x^3 - 2x^2 + 3x + 1}{(x^2 - x - 2)(x^2 + 1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)} + \frac{Cx + D}{(x^2+1)}$$

$$2x^3 - 2x^2 + 3x + 1 = A(x+1)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x+1)(x-2)$$

Equate coefficients:

$$x^3 : 2 = A + B + C$$

$$x^2 : -2 = A - 2B + D - C$$

$$x^1 : 3 = A + B - 2C - D$$

$$x^0 : 1 = A - 2B - 2D$$

From x^3: $A = 2 - B - C$ so the other equations become

$$x^2 : \quad -2 = 2 - B - C - 2B + D - C$$

$$-4 = -3B - 2C + D$$

$$x^1 : \quad 3 = 2 - B - C + B - 2C - D$$

$$1 = -3C - D$$

$$x^0 : \quad 1 = 2 - B - C - 2B - 2D$$

$$-1 = -3B - 2D$$

From x^1: $D = -3C - 1$ so

$$x^2 : \quad -4 = -3B - 2C + -3C - 1$$

$$-3 = -3B - 5C$$

$$x^0 : \quad -1 = -3B - 2(-3C - 1)$$

$$-3 = -3B + 6C$$

It follows that $C = 0$ and $B = 1$, whence $D = -1$ and $A = 1$.

Plug in:

$x = 2 : 16 - 8 + 6 + 1 = A(3)(5)$ or $15 = 15A$ or $A = 1$.

$x = -1 : -2 - 2 - 3 + 1 = B(-3)(2)$ or $-6 = (-6)B$ or $B = 1$.

$x = 0 : 1 = A + (-2)B + (-2)D$ or $1 = -1 + (-2)D$ or $D = -1$.

$x = 1 : 2 - 2 + 3 + 1 = A(2)(2) + B(-1)(2) + (C + D)(2)(-1)$ or $4 = 4 - 2 - 2(C - 1)$ or $2 = 2 - 2C$ or $C = 0$.

Math 126, Quiz #7
March 27, 2001

Which improper integrals below converge and which diverge? A brief indication of your reasoning should be given.

a) $\displaystyle\int_0^\infty e^{-x^3}\,dx$

b) $\displaystyle\int_0^\infty \frac{1}{\sqrt[3]{x^2+1}}\,dx$

Solution

a) $\displaystyle\int_0^\infty e^{-x^3}\,dx$ converges if and only if $\displaystyle\int_1^\infty e^{-x^3}\,dx$ converges.

On the interval $[1,\infty)$, $x \geq 1$ so $x^2 \geq 1$ and $x^3 \geq x$ so $e^x \leq e^{x^3}$ so $e^{-x^3} \leq e^{-x}$. Now $\displaystyle\int_1^\infty e^{-x}\,dx$ converges since $\displaystyle\lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty} -e^{-x}\Big|_1^t = e^{-1} - \lim_{t\to\infty} e^{-t} = e^{-1} - 0$. By the first comparison test for improper integrals, $\displaystyle\int_1^\infty e^{-x^3}\,dx$ converges and hence so does $\displaystyle\int_0^\infty e^{-x^3}\,dx$.

b) Roughly speaking $\dfrac{1}{\sqrt[3]{x^2+1}}$ behaves near ∞ like $x^{-2/3}$. More precisely,

$\displaystyle\lim_{x\to\infty}\frac{\frac{1}{\sqrt[3]{x^2+1}}}{x^{-2/3}} = \lim_{x\to\infty}\frac{x^{2/3}}{\sqrt[3]{x^2+1}} = \lim_{x\to\infty}\frac{1}{\sqrt[3]{1+\frac{1}{x^2}}} = 1$. Since $0 < 1 < \infty$ we would like to use the limit comparison test, comparing the integral for $\dfrac{1}{\sqrt[3]{x^2+1}}$ with the one for $x^{-2/3}$. Annoyingly, $x^{-2/3}$ has an additional singularity at 0 so we proceed as follows.

$\displaystyle\int_0^\infty \frac{1}{\sqrt[3]{x^2+1}}\,dx$ converges if and only if $\displaystyle\int_1^\infty \frac{1}{\sqrt[3]{x^2+1}}\,dx$ converges and by the limit comparison test for improper integrals, $\displaystyle\int_1^\infty \frac{1}{\sqrt[3]{x^2+1}}\,dx$ converges if and only if $\displaystyle\int_1^\infty x^{-2/3}\,dx$ converges.

But $\displaystyle\int_1^\infty x^{-2/3}\,dx = \lim_{t\to\infty}\frac{x^{1/3}}{1/3}\Big|_1^t \lim_{t\to\infty} 3 - 3t^{1/3} = 3 - \infty$ so $\displaystyle\int_1^\infty x^{-2/3}\,dx$ diverges and hence so does $\displaystyle\int_0^\infty \frac{1}{\sqrt[3]{x^2+1}}\,dx$.
