13. Find the area under the region bounded by the curves $y = 2^x$ and $y = 5^x$, between $x = 0$ and $x = 1$.

. . . .

Since $5^x > 2^x$ for $x > 0$ the requested area is

$$\int_0^1 5^x \;-\; 2^x \, dx = \frac{5^x}{\ln 5} - \frac{2^x}{\ln 2}\Bigg|_0^1 = \left(\frac{5}{\ln 5} - \frac{2}{\ln 2}\right) - \left(\frac{1}{\ln 5} - \frac{1}{\ln 2}\right) = \left(\frac{4}{\ln 5} - \frac{1}{\ln 2}\right).$$

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14. Evaluate the limit

$$\lim_{x \to 0^+} (\sin x)^{\tan x}$$

. . . .

Let $y = \lim_{x \to 0^+} (\sin x)^{\tan x}$. Then

$\ln y = \lim_{x \to 0^+} \tan x \, \ln(\sin x) = \lim_{x \to 0^+} \dfrac{\ln(\sin x)}{\cot x}.$

When $x \to 0^+$, $\tan(x) \to 0$, $\cot(x) \to \infty$ and $\ln(\sin x)) \to -\infty$.

This is an indeterminate form where we can apply L'Hospital's rule, so

$\ln y = \lim_{x \to 0^+} \dfrac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x \to 0^+} -\sin x \cos x = 0.$

Since $\ln y = 0$, $y = e^0 = 1$.

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15. Use logarithmic differentiation to find the derivative of

$$y = \frac{(x^2+1)^{2.2}(x+1)^{1.2}}{(x^3-1)^{0.3}}$$

. . . .

This is a logarithmic differentiation problem, so start by taking the log of both sides and simplifying the right hand side.

$$\begin{aligned} \ln\, y &= \ln\left(\frac{(x^2+1)^{2.2}(x+1)^{1.2}}{(x^3-1)^{0.3}}\right) \\ &= \ln\big((x^2+1)^{2.2}\big) + \ln\big((x+1)^{1.2}\big) - \ln\big((x^3-1)^{0.3}\big) \\ &= 2.2\ln(x^2+1) + 1.2\ln(x+1) - 0.3\ln(x^3-1) \end{aligned}$$

Differentiating both sides with respect to x we get

$$\frac{y'}{y} = 2.2\frac{2x}{x^2+1} + 1.2\frac{1}{x+1} - 0.3\frac{3x^2}{x^3-1} = \frac{4\,x}{x^2+1} + \frac{1.2}{x+1} - \frac{0.9\,x^2}{x^3-1}$$

Now solve for y'

$$\frac{dy}{dx} = \left(\frac{(x^2+1)^{2.2}(x+1)^{1.2}}{(x^3-1)^{0.3}}\right) \cdot \left(\frac{4\,x}{x^2+1} + \frac{1.2}{x+1} - \frac{0.9\,x^2}{x^3-1}\right)$$
