13. Find the center of mass (centroid) of the region bounded by the curves $y = \cos x$, $y = 0$, $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.
You may use symmetry as part of the justification for your answer.

. . . .

First we find the area $A = \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\ dx$. The alternative of $A = \displaystyle\int_0^1 2\arccos(y)\ dy$ does not look as easy. Then $M_x = \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(x)}{2}\ \cos(x)\ dx$ and $M_y = \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\ \cos(x)\ dx$. By symmetry, $M_y = 0$: the integral can also be done by parts. $A = \sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1-(-1) = 2$. $M_x = \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2(x)}{2}\ dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2x)}{4}\ dx = \frac{x}{4} + \frac{\sin(2x)}{8}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \left(\frac{\pi}{8}+0\right) - \left(\frac{-\pi}{8}+0\right) = \frac{\pi}{4}$.

Hence the coordinates of the center of mass are at $\left(\dfrac{M_y}{A}, \dfrac{M_x}{A}\right) = \left(0, \dfrac{\pi}{8}\right)$.

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14. Solve the initial value problem

$$\begin{cases} xy' + xy + y = e^{-x} \\ y(1) = \dfrac{2}{e} \end{cases}$$

. . . .

This is a linear differential equation. First we bring the equation to the standard form by dividing by x. We get:

$$\frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = e^{-x}/x$$

We identify $P(x) = (1 + \frac{1}{x})$ and $Q(x) = \frac{e^{-x}}{x}$.
We define

$$I(X) = e^{\int P(x)dx} = e^{\int 1+\frac{1}{x}dx} = e^{x+lnx} = e^x\ x\ .$$

The solution for the differential equation is:

$$y = \frac{1}{xe^x}\int e^x x\ e^{-x}/xdx = \frac{1}{xe^x}\int dx = \frac{1}{xe^x}(1+C)\ .$$

Evaluating at $x = 1$, we get $\frac{2}{e} = y(1) = \frac{1+C}{e}$,
which implies $C = 1$,l so the answer is

$$y = \frac{2}{x\ e^x}$$

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15. Determine whether or not the improper integral $\displaystyle\int_1^{+\infty} \frac{2\cos(x^2)+100}{x^{5/4}}\,dx$ is convergent. To receive credit for this problem you must justify your answer.

. . . .

We have no hope of evaluating the indefinite integral $\displaystyle\int \frac{2\cos(x^2)+100}{x^{5/4}}\,dx$ directly so we turn to a comparison theorem. Note first that $-1 \le \cos(x^2) \le 1$ so

$$0 < \frac{98}{x^{5/4}} \le \frac{2\cos(x^2)+100}{x^{5/4}} \le \frac{102}{x^{5/4}}$$

for $x \ge 1$. Hence we can compare our integral with either $\displaystyle\int_1^{+\infty} \frac{98}{x^{5/4}}\,dx$ or $\displaystyle\int_1^{+\infty} \frac{102}{x^{5/4}}\,dx$. But $\displaystyle\int_1^{+\infty} \frac{98}{x^{5/4}}\,dx = 98 \int_1^{+\infty} \frac{1}{x^{5/4}}\,dx$ and $\displaystyle\int_1^{+\infty} \frac{102}{x^{5/4}}\,dx = 102 \int_1^{+\infty} \frac{1}{x^{5/4}}\,dx$ so it suffices to evaluate $\displaystyle\int_1^{+\infty} \frac{1}{x^{5/4}}\,dx$. Since $\frac{5}{4} > 1$ this integral converges, hence does the original one.

In more detail, $\displaystyle\int_1^{+\infty} \frac{1}{x^{5/4}}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^{5/4}}\,dx = \lim_{t\to\infty} \left.\frac{x^{-5/4+1}}{-5/4+1}\right|_1^t = \lim_{t\to\infty} \left.\frac{-4}{x^{1/4}}\right|_1^t = 4 - \lim_{t\to\infty} \frac{1}{t^{1/4}} = 4$ so $\displaystyle\int_1^{+\infty} \frac{1}{x^{5/4}}\,dx$ converges.