13. Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^n \, \frac{n^2+1}{n^3-9}$$

is absolutely convergent, conditionally convergent or divergent.

. . . .

We first check for absolute convergence. In other words, does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3-9}$ converge or not? Use the Limit Comparison Test to compare our series to the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ which we know diverges. Compute $\displaystyle\lim_{n\to\infty} \frac{\frac{n^2+1}{n^3-9}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n(n^2+1)}{n^3-9}$. This is a rational function with the degree of the numerator (3) equal to the degree of the denominator. Hence the limit is the ration of the degree 3 coefficients or $\frac{1}{1} = 1$. Since $0 < 1 < \infty$, the Limit Comparison Test shows $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3-9}$ divergses.

The series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \, \frac{n^2+1}{n^3-9}$ is alternating and $\displaystyle\lim_{n\to\infty} \frac{n^2+1}{n^3-9} = 0$ since it is a limit of a rational function for which the degree of the numerator is less than the degree of the denominator. Finally, since $\displaystyle\frac{d\,\frac{x^2+1}{x^3-9}}{dx} = \frac{2x(x^3-9)-(x^2+1)(3x^2)}{(x^3-9)^2} = \frac{-x^4-3x^2-18x}{(x^3-9)^2} < 0$ for $x > 0$, the sequence $\displaystyle\frac{n^2+1}{n^3-9}$ is decreasing. Hence the Alternating Series Test shows $\displaystyle\sum_{n=1}^{\infty} (-1)^n \, \frac{n^2+1}{n^3-9}$ is convergent and therefore conditionally convergent.

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14. Find the radius of convergence and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \, (x-3)^n$$

. . . .

Begin with the radius using either the Ratio method or the Root method. For the Ration method, compute

$$\lim_{n\to\infty} \frac{\frac{|x-3|^{n+1}}{\sqrt{n+1}}}{\frac{|x-3|^n}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \cdot |x-3| = |x-3|.$$

For the Root method compute

$$\lim_{n\to\infty} \sqrt[n]{\frac{|x-3|^n}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\sqrt{n}}} \cdot |x-3| = \frac{1}{\sqrt{\lim\limits_{n\to\infty} \sqrt[n]{n}}} \cdot |x-3| = |x-3|.$$

Either of these calculations shows that the radius of convergence is 1.

To calculate the interval of convergence we need to evaluate the two series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\,(1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\,(-1)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

The second series is a p–series with $p = 1/2 \leq 1$ and so diverges. The first series is an alternating p–series with $p = 1/2 > 0$ and so converges. Hence the interval of convergence is $(2, 4]$.

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15. Find the power series centered at the origin for $f(x) = \frac{1}{(2-x)^2}$.
Hint: $f(x)$ is related to the derivative of a series you should know.

. . . .

First note that $\dfrac{d\,\frac{1}{2-x}}{dx} = \dfrac{-1}{(2-x)^2} \cdot (-1) = \dfrac{1}{(2-x)^2}$. Next recall that you can figure out a power series for $\dfrac{1}{2-x} = \dfrac{1}{2} \cdot \dfrac{1}{1-(x/2)} = \dfrac{1}{2} \cdot \displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$. Taking derivatives we see $\dfrac{1}{(2-x)^2} = \displaystyle\sum_{n=1}^{\infty} \frac{n\,x^{n-1}}{2^{n+1}}$, or if you prefer $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)\,x^n}{2^{n+2}}$.
