Exam I
February 15, 2001

11.

(a) First we need to determine the corner points. They are $(\frac{1}{4}, 4)$, $(1, 4)$, $(4, 1)$ and $(4, \frac{1}{4})$. It is also good to note at this point that the integral will need to be split into two pieces. We also note that we may take the density to be 1 in this problem. Hence if we integrate along the x–axis we get for the area (= mass)

$$\begin{aligned}
\text{Area} = & \int_{\frac{1}{4}}^{1} \left(4 - \frac{1}{x}\right) dx + \int_{1}^{4} \left(\frac{4}{x} - \frac{1}{x}\right) dx \\
= & \big[4x - \ln x\big]_{\frac{1}{4}}^{1} + \big[3 \ln x\big]_{1}^{4} \\
= & \big(4 - \ln 1\big) - \big(1 - \ln \frac{1}{4}\big) + 3 \ln 4 - 3 \ln 1 \\
= & 3 + \ln \frac{1}{4} + 3 \ln 4 = 3 - \ln 4 + 3 \ln 4 = 3 + 2 \ln 4
\end{aligned}$$

If you preferred to integrate up the y–axis you get an identical formula except that everywhere you see an x above, put a y.

(b) You have four choices. Do you want to compute the moment about the x–axis or the y–axis and do you want to integrate along the x–axis or the y–axis? To compute the moment about the y–axis and integrate along the x–axis, proceed as follows.

$$\begin{aligned}
\text{Moment}_y = & \int_{\frac{1}{4}}^{1} \underbrace{x}_{\substack{\text{dist.} \\ \text{from} \\ y\text{–axis}}} \underbrace{\left(4 - \frac{1}{x}\right) dx}_{\text{mass of rect.}} + \int_{1}^{4} x \left(\frac{4}{x} - \frac{1}{x}\right) dx \\
= & \big[2x^2 - x\big]_{\frac{1}{4}}^{1} + \big[3x\big]_{1}^{4} = 2 - 1 - \frac{2}{16} + \frac{1}{4} + 12 - 3 = \frac{81}{8}
\end{aligned}$$

If you choose instead to calculate the moment about the x–axis while still integrating along the x–axis you would proceed as follows.

$$\text{Moment}_x = \int_{\frac{1}{4}}^{1} \underbrace{\frac{1}{2}\left(4+\frac{1}{x}\right)}_{\substack{\text{dist. from}\\ \text{center of mass}\\ \text{to } x\text{–axis}}} \underbrace{\left(4-\frac{1}{x}\right)}_{\text{mass of rect.}} dx \quad + \int_{1}^{4} \frac{1}{2}\left(\frac{4}{x}+\frac{1}{x}\right)\left(\frac{4}{x}-\frac{1}{x}\right) dx$$

$$= \int_{\frac{1}{4}}^{1} 8 - \frac{1}{2x^2}\, dx + \int_{1}^{4} \frac{15}{2x^2}\, dx$$

$$= \left[8x + \frac{1}{2x}\right]_{\frac{1}{4}}^{1} + \left[\frac{-15}{2x}\right]_{1}^{4} = \left(8+\frac{1}{2}\right) - \left(2+2\right) + \left(-\frac{15}{8} - \left(\frac{-15}{2}\right)\right)$$

$$= 4 + \frac{4}{8} + \frac{-15}{8} + \frac{60}{8} = \frac{32}{8} + \frac{4}{8} + \frac{-15}{8} + \frac{60}{8} = \frac{96-15}{8} = \frac{81}{8}$$

If you want the integrals for integrating along the y–axis, just switch x and y in the formulae above.

(c) By symmetry, the moment about the x–axis is the same as the moment about the y–axis and the center of mass has equal x and y coordinates, each one being

$$\frac{\frac{81}{8}}{3 + 2\ln 4}$$

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12. Complete the square: $x^2 + 2x + 2 = (x+1)^2 + 1$. This suggests the trig. substitution $\tan\theta = x+1$: $dx = \sec^2\theta\, d\theta$ and $x^2 + 2x + 2 = (x+1)^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$. When $x = -1$, $x + 1 = 0 = \tan\theta$, so $\theta = 0$; when $x = 0$, $x + 1 = 1 = \tan\theta$, so $\theta = \frac{\pi}{4}$. Hence

$$\int_{-1}^{0} \frac{1}{x^2+2x+2}\, dx = \int_{0}^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^2\theta}\, d\theta = \int_{0}^{\frac{\pi}{4}} d\theta = \theta\Big|_{0}^{\frac{\pi}{4}} = \frac{\pi}{4} - 0 = \frac{\pi}{4}\ .$$

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13. Separating variables gives $\dfrac{dy}{\sqrt{1-y^2}} = x$: integrating gives

$$\arcsin(y) = \frac{x^2}{2} + C.$$

Now since $y(0) = 0$ we see that the constant $C = 0$.
Hence the unique function is

$$y = \sin(\frac{x^2}{2})$$

and $y(1) = \sin(\frac{1}{2})$.

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14. To find the general solution to the differential equation

$$y' \; = \; \frac{2y}{x} + x \quad (x > 0)$$

first put it into standard form:

$$y' + \frac{-2}{x} \cdot y = x$$

so $P(x) = \frac{-2}{x}$ and $Q(x) = x$. Then find the integrating factor $v = e^{\int P(x)\,dx}$. $\displaystyle\int \frac{-2}{x} = -2\ln x + C$ so we may take $v(x) = e^{-2\ln x}$ which is $v(x) = x^{-2}$. Then we do the second integral $\displaystyle\int v(x)\,Q(x)\,dx = \int x^{-2} \cdot x\,dx = \int x{-1}\,dx = \ln x + C$. Since we are told $x > 0$, there is no need for absolute values in the $\ln x$. Finally,

$$y(x) = x^{-2}\big(\ln x + C\big)$$

is the general solution.
