

Math 126
Exam II
March 20, 2001

9. First by long division the integrand is equal to

$$2x + \frac{1}{x(x-1)}.$$

Expand the second term using partial fractions:

$$\frac{1}{x(x-1)} = \frac{A}{x-1} + \frac{B}{x}$$

or

$$1 = Ax + B(x-1).$$

Solving the equations gives

$$\frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x}.$$

We can then use standard integration formulas to get the general antiderivative

$$x^2 + \ln|x-1| - \ln|x| + C.$$

10. Use the substitution $x = \frac{3}{4} \sin \theta$: then $dx = \frac{3}{4} \cos \theta d\theta$ and

$$\begin{aligned} \int \frac{x^2}{\sqrt{9-16x^2}} &= \left(\frac{3}{4}\right)^2 \int \frac{\sin^2 \theta \cos \theta d\theta}{\sqrt{9-9\sin^2 \theta}} \\ &= \frac{9}{64} \int \sin^2 \theta d\theta \\ &= \frac{9}{128} (\theta - \sin \theta \cos \theta) + C \quad (\text{from memory or see below}) \\ &= \frac{9}{128} \left(\arcsin \frac{4x}{3} - \frac{4x}{3} \sqrt{1 - \left(\frac{4x}{3}\right)^2} \right) + C \\ &= \frac{9}{128} \left(\arcsin \frac{4x}{3} - \frac{4x}{9} \sqrt{9-16x^2} \right) + C \end{aligned}$$

The integral $\int \sin^2 \theta d\theta$ can be computed several different ways if you didn't memorize the answer.

1. Use the trig. identities $\sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta))$ and $\sin(2\theta) = 2 \sin \theta \cos \theta$: $\int \sin^2 \theta d\theta =$

$$\int \frac{1}{2}(1 - \cos(2\theta)) d\theta = \frac{1}{2}(\theta - \frac{1}{2} \sin(2\theta)) + C = \frac{1}{2}(\theta - \sin \theta \cos \theta) + C$$

2. Use integration by parts: $u = \sin \theta$, $dv = \sin \theta d\theta$;

$$\int \sin^2 \theta d\theta = -\sin \theta \cos \theta + \int \cos^2 \theta d\theta = -\sin \theta \cos \theta + \int 1 - \sin^2 \theta d\theta = -\sin \theta \cos \theta + \theta - \int \sin^2 \theta d\theta \text{ so } 2 \int \sin^2 \theta d\theta = \theta - \sin \theta \cos \theta + C.$$

11.

$$\int e^{\sqrt{x}} dx = \int 2u e^u du = 2ue^u - \int 2e^u du = 2ue^u - 2e^u + C$$

Substitution $u = \sqrt{x}$ $du = \frac{1}{2\sqrt{x}} dx$ $2\sqrt{x} du = dx$ $2u du = dx$	Parts $w = 2u$ $dv = e^u du$ $dw = 2du$ $v = e^u$
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$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

This problem CAN be done via integration by parts, but it is so weird that no one actually did it this way. Still

$$\int e^{\sqrt{x}} dx = \int \underbrace{\sqrt{x}}_u \underbrace{\frac{1}{\sqrt{x}} e^{\sqrt{x}} dx}_{dv} = 2\sqrt{x} e^{\sqrt{x}} - \int \frac{1}{\sqrt{x}} e^{\sqrt{x}} dx$$

$$du = \frac{1}{2\sqrt{x}} ; v = 2e^{\sqrt{x}}$$

$$= 2\sqrt{x} e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

12. Since the answer is not obvious, rewrite:

$$\lim_{x \rightarrow 0} (1 + 2x)^{1/x} = e^{\lim_{x \rightarrow 0} \ln(1 + 2x)^{1/x}}$$

and we concentrate on $\lim_{x \rightarrow 0} \ln(1 + 2x)^{1/x} = \lim_{x \rightarrow 0} \frac{\ln(1 + 2x)}{x}$.

Since $\ln(1 + 2 \cdot 0) = \ln(1) = 0$ this last limit is of the form $\frac{0}{0}$ so we may apply l'Hôpital's rule. This requires us to compute

$$\lim_{x \rightarrow 0} \frac{\frac{d \ln(1+2x)}{dx}}{\frac{dx}{dx}} = \lim_{x \rightarrow 0} \frac{\frac{2}{1+2x}}{1} = 2 .$$

Hence

$$\lim_{x \rightarrow 0} (1 + 2x)^{1/x} = e^2 .$$

Another way to go, which no one completed successfully, is to *remember*

$\lim_{t \rightarrow \infty} \left(1 + \frac{a}{t}\right)^t = e^a$ and proceed as follows. First compute $\lim_{x \rightarrow 0^+} (1 + 2x)^{1/x}$ by letting $x = \frac{1}{t}$

and $\lim_{x \rightarrow 0^+} (1 + 2x)^{1/x} = \lim_{t \rightarrow \infty} \left(1 + \frac{2}{t}\right)^t = e^2$. After this, you know $\lim_{x \rightarrow 0} (1 + 2x)^{1/x} = e^2$ OR

it does not exist depending on what happens with $\lim_{x \rightarrow 0^-} (1 + 2x)^{1/x} = \lim_{t \rightarrow -\infty} \left(1 + \frac{2}{t}\right)^t$. If

we let $T = -t$, then $\lim_{t \rightarrow -\infty} \left(1 + \frac{2}{t}\right)^t = \lim_{T \rightarrow \infty} \left(1 + \frac{-2}{T}\right)^{-T} = \frac{1}{\lim_{T \rightarrow \infty} \left(1 + \frac{-2}{T}\right)^T} = \frac{1}{e^{-2}} = e^2$

so $\lim_{x \rightarrow 0^-} (1 + 2x)^{1/x} = e^2$ and $\lim_{x \rightarrow 0} (1 + 2x)^{1/x} = e^2$.
