Math 126
Exam III
April 24, 2001

9. Does

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^{2n})}$$

converge or diverge? Show your reasoning and state clearly any theorems or tests you are using.

There are many solutions. Easiest is probably the root test: $\sum a_n$ with $a_n = \dfrac{(n!)^n}{n^{2n}}$.
$\sqrt[n]{a_n} = \dfrac{n!}{n^2} \to \infty \quad$ as $n \to \infty$.
As this limit ∞ is > 1, the root test tells you that the series $\sum a_n$ diverges.
To see why $\displaystyle\lim_{n\to\infty} \frac{n!}{n^2} = \infty$, write it as

$$\lim_{n\to\infty} \frac{n(n-1)\cdot(n-2)!}{n^2} = \lim_{n\to\infty} \frac{n-1}{n} \cdot \lim_{n\to\infty} (n-2)! = 1 \cdot \infty$$

It was also OK to have memorized that $\displaystyle\lim_{n\to\infty} \frac{n!}{n^k} = \infty$ for every k.

Do NOT use l'Hôpital's rule on $\displaystyle\lim_{n\to\infty} \frac{n!}{n^2}$ since you can not differentiate $n!$.

Do NOT use the ratio test on $\displaystyle\lim_{n\to\infty} \frac{n!}{n^2}$. This test would not tell you anything about the *sequence* $\left\{\dfrac{n!}{n^2}\right\}$. Rather, it tells you about the *series* $\sum \dfrac{n!}{n^2}$ (with which you have no business here).

Second solution: Some people looked at the *series* $\sum \dfrac{n!}{n^2}$ and determined that it diverged using the ratio test: $\dfrac{\frac{(n+1)!}{(n+1)^2}}{\frac{n!}{n^2}} = \dfrac{n^2}{n+1} \to \infty$ as $n \to \infty$.

By direct comparison, $\dfrac{(n!)^n}{n^{2n}} > \dfrac{n!}{n^2}$ whenever $\dfrac{n!}{n^2} > 1$ which happens for $n > 3$.

Third solution: Once you have seen $\displaystyle\lim_{n\to\infty} \frac{n!}{n^2} = \infty$ and its immediate consequence $\displaystyle\lim_{n\to\infty} \frac{(n!)^n}{n^{2n}} = \infty$ you can use the fact that $\displaystyle\lim_{n\to\infty} a_n \neq 0$ to determine that the series diverges.

Fourth solution: The messiest way to proceed is via the ratio test, but it can be done reasonably well.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\big((n+1)!\big)^{n+1} n^{2n}}{(n+1)^{2(n+1)}(n!)^n} = \frac{(n+1)^{n+1}(n!)^{n+1}n^{2n}}{(n+1)^{2n+2}(n!)^n} = \frac{n^{2n}}{(n+1)^{n+1}} \cdot n! \\ &= \left(\frac{n^2}{n+1}\right)^n \cdot \frac{n!}{n+1} = \left(\frac{n^2}{n+1}\right)^n \cdot \frac{n}{n+1} \cdot (n-1)!\end{aligned}$$

The term $\dfrac{n}{n+1} \to 1$ as $n \to \infty$ and both $\left(\dfrac{n^2}{n+1}\right)^n \to \infty$ and $(n-1)! \to \infty$ as $n \to \infty$. Hence $\frac{a_{n+1}}{a_n} \to \infty > 1$ and therefore the series diverges.

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10. Does the integral

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$$

converge or diverge?
We compare the function $\frac{1}{\sqrt{x}(x+1)}$ with $\frac{1}{x^{\frac{3}{2}}}$
$\displaystyle\lim_{x\to\infty} \frac{x^{\frac{3}{2}}}{\sqrt{x}(x+1)} = \lim_{x\to\infty} \frac{x^{\frac{3}{2}}}{x^{\frac{3}{2}} + x^{\frac{1}{2}}} = 1$
This implies that the integral above converges if and only if $\int_0^\infty \frac{dx}{x^{\frac{3}{2}}}$ converges. But $\displaystyle\lim_{b\to\infty} \int_0^b \frac{dx}{x^{\frac{3}{2}}} = \lim_{b\to\infty} (-2)[x^{-\frac{1}{2}}]_1^b = \lim_{b\to\infty} [2 - 2\frac{1}{\sqrt{b}}] = 2$, so it converges and so does the original integral.

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11. Find the **interval** of convergence of the series $\sum_{n=1}^{\infty} \frac{x^{2n}}{2^n\, n^2}$.

First calculate the radius of convergence. Use either the nth root test or the ratio test:

$$\lim_{n\to\infty} \sqrt[n]{\frac{x^{2n}}{2^n\, n^2}} = \lim_{n\to\infty} \frac{x^2}{2\, \sqrt[n]{n^2}} = \lim_{n\to\infty} \frac{x^2}{2(\sqrt[n]{n})^2} = \frac{x^2}{2}$$

OR

$$\lim_{n\to\infty} \frac{\frac{x^{2(n+1)}}{2^{n+1}\,(n+1)^2}}{\frac{x^{2n}}{2^n\, n^2}} = \lim_{n\to\infty} \frac{x^2}{2} \cdot \frac{n^2}{(n+1)^2} = \frac{x^2}{2}$$

Hence $-1 < \frac{x^2}{2} < 1$ or $-2 < x^2 < 2$. Several people had issues with going from x^2 to x. It is true that x^2 can not be negative so $-2 < x^2 < 2$ can be replaced with $0 \le x^2 < 2$ *but* $x^2 < 2$ constrains x from being too negative. In particular, $x^2 < 2$ implies $-\sqrt{2} < x < \sqrt{2}$.

We now need to check the endpoints: the two series in question are

$$\sum_{n=1}^{\infty} \frac{(\pm\sqrt{2})^{2n}}{2^n\, n^2} = \sum_{n=1}^{\infty} \frac{2^n}{2^n\, n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \ .$$

This latter series is a p–series with $p = 2$ and therefore is convergent. Hence the interval of convergence is $[-\sqrt{2}, \sqrt{2}]$.

You were not asked, but we know that the convergence is absolute on the entire interval $[-\sqrt{2}, \sqrt{2}]$: it converges absolutely on $(-\sqrt{2}, \sqrt{2})$ by our theory and it converges absolutely at the endpoints because that is what we just shown.

12.
(a) Show that

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}$$

provided that $|x| < 1$.
(b) Find

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}.$$

For part (a), this is a geometric series with first term 1 and ratio $-x^2$. Hence, provided $|x| < 1$ the series converges to

$$\frac{1}{1+x^2}.$$

For part (b), we can integrate both sides and use the term-by-term integration theorem for the series on the left. This gives

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = \arctan(x) + C.$$

The constant C must be zero as we see from setting $x = 0$.

Now we plug in $x = \frac{1}{\sqrt{3}} < 1$ to both sides and find

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}} = \arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}.$$
