Math. 126 Quiz #11 April 30, 2002

The MacLaurin series for $\cos x$ is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

a. What is the radius of convergence of this series? (Just an answer is sufficient - no reason need be given.)

b. Write down the MacLaurin series for $\frac{\cos x - 1}{x^2}$. Give a reason why it is the MacLaurin series.

c. Write down the MacLaurin series for the function $\int_0^x \cos t^2 \, dt$.

. . . .

For (a), the radius of convergence is ∞, a result you should memorize. To see why (which you were NOT asked to do), compute $\lim_{n\to\infty} \frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!)} \cdot |x|^2 = \lim_{n\to\infty} \frac{|x|^2}{(2n+1)(2n+2)} = 0$.

For (b), $\cos x - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ and hence $\frac{\cos x - 1}{x^2} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n)!}$. This is equal to $\sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^{2n}}{(2n+2)!}$. Why is this the MacLaurin series for $\frac{\cos x - 1}{x^2}$? The best answer is that by the calculations we just did, the series represents the function. The MacLaurin series for the power series is itself (that's a theorem) and since the series and the function are the same, the series is also the MacLaurin series for the function.

For (c), $\cos t^2 = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{(2n)!}$, so $\int_0^x \cos t^2 \, dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{(2n)!} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}$.

Math. 126 Quiz #10 April 16, 2002

Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^2+n}}$ converges by *checking* the hypotheses of the Alternating Series Test.

Then show the calcualtions needed to find an m such that

$$0 < \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^2+n}} - \sum_{n=1}^{m} \frac{(-1)^{n+1}}{\sqrt{n^2+n}} < 0.01$$

. . . .

To see that the series converges using the Alternating Series Test, we need to show that $\lim_{n\to\infty} b_n = 0$ and that $b_{n+1} \le b_n$. First, note that

$$b_n = \frac{1}{\sqrt{n^2+n}}.$$

Since

$$\lim_{n\to\infty} \sqrt{n^2+n} = \infty,$$

we have that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+n}} = 0.$$

To check that $b_{n+1} \le b_n$, ask: Is

$$\frac{1}{\sqrt{(n+1)^2+n+1}} \le \frac{1}{\sqrt{n^2+n}}?$$

After clearing fractions and squaring, we see that the inequality holds if

$$n^2+n \le (n+1)^2+n+1$$

holds. This second inequality is clearly true. (Note: another way is to show that $\frac{d}{dn}b_n = -\frac{1}{2}(n^2+n)^{-\frac{3}{2}}(2n+1)$ and this is clearly < 0 for $n \ge 1$.)

Recall that $|R_m| \le b_{m+1}$. Thus, for the error calculation, we need to solve

$$b_{m+1} < 0.01\ .$$

(Also, since

$$0 < \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^2+n}} - \sum_{n=1}^{m} \frac{(-1)^{n+1}}{\sqrt{n^2+n}},$$

note that m must be odd since the remainder has the same sign as b_{m+1}.)

Math. 126 Quiz #9 April 9, 2002

For each of the series below, do two things. First compute $\lim\limits_{n\to\infty} a_n$ and then use this calculation to say if you are **certain** that $\sum\limits_{n=1}^{\infty} a_n$ diverges or if the limit calculation does not suffice to say if the series converges or diverges. Just circle **diverges** or **can not tell** after the series for the second part of each question

A. $a_n = \dfrac{1}{n\,!}$:

Recall $n\,! = n(n-1)\cdots 2 \cdot 1$

a. $\lim\limits_{n\to\infty} a_n =$

b. $\sum\limits_{n=1}^{\infty} a_n$ **diverges** **can not tell**

B. $a_n = \dfrac{n}{1+\sqrt{n}}$:

a. $\lim\limits_{n\to\infty} a_n =$

b. $\sum\limits_{n=1}^{\infty} a_n$ **diverges** **can not tell**

. . . .

For A we note $0 < \frac{1}{n\,!} = \frac{1}{n} \cdot \frac{1}{n-1} \cdots \frac{1}{2} \cdot \frac{1}{1} < \frac{1}{n}$ and since $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, the Squeeze Theorem promises $\lim\limits_{n\to\infty} \dfrac{1}{n\,!} = 0$. This tells us nothing about whether the series $\sum\limits_{n=1}^{\infty} \frac{1}{n\,!}$ converges. (Remark: It is correct that the series does converge, in fact to $e-1$, but this calculation does not show it.)

For B we write $\dfrac{n}{1+\sqrt{n}} = \dfrac{\sqrt{n}}{\frac{1}{\sqrt{n}}+1}$ so $\lim\limits_{n\to\infty} \dfrac{n}{1+\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{\frac{1}{\sqrt{n}}+1} = \dfrac{\infty}{0+1} = \infty$. The Divergence Test shows that the series $\sum\limits_{n=1}^{\infty} \frac{n}{1+\sqrt{n}}$ diverges.

Math. 126 Quiz #8 April 2, 2002

Write an equation for the hyperbola which has foci at $(3,3)$ and $(3,11)$ and which has asymptotes $y = x + 4$ and $y = 10 - x$. What are the coordinates of the vertices?

. . . .

The center is halfway between the foci so has coordinates $(3,7)$. Thus $c = 4$ since it is the distance from the center to a focus.

The vertices of the hyperbola are on the line $x = 3$ since they lie on the line between the foci, so the equation has the form

$$\frac{(y-7)^2}{a^2} - \frac{(x-3)^2}{b^2} = 1$$

where $16 = c^2 = a^2 + b^2$.

The asymptotes have slope ± 1 from the slope–intercept form of the equation. The slopes are $\pm\frac{a}{b}$ for any hyperbola so $a = b$ and $16 = 2a^2$ so $a^2 = b^2 = 8$.

The vertices are a distance a from the center. Since $a^2 = 8$, $a = 2\sqrt{2}$ and the vertices are at $(3, 7 \pm 2\sqrt{2}\ )$.

Math. 126 Quiz #7 March 26, 2002

1. Write down a definite integral which gives the length of the curve $x(t) = t^3 + 4t + 1$, $y(t) = t^4 + t^2$ for $0 \le t \le 5$. Do **NOT** attempt to evaluate either this integral or the integrals you will write down in parts 2 and 3.
2. Write down a definite integral which gives the surface area of the surface of revolution obtained by rotating the parameterized curve in 1 around the x–axis.
3. Write down a definite integral which gives the surface area of the surface of revolution obtained by rotating the parameterized curve in 1 around the y–axis.

. . . .

We have the curve $x(t) = t^3 + 4t + 1$, $y(t) = t^4 + t^2$ for $0 \le t \le 5$.

Note $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \sqrt{(3t^2+4)^2 + (4t^3+2t)^2}\ dt$

1. The definite integral which gives the length of this curve is

$$L = \int_0^5 \sqrt{(3t^2+4)^2 + (4t^3+2t)^2}\ dt\ .$$

2. The definite integral which gives the surface area of the surface of revolution obtained be rotating the parameterized curve around the x-axis is

$$S = \int_0^5 2\pi(t^4+t^2)\sqrt{(3t^2+4)^2 + (4t^3+2t)^2}\ dt$$

since $t^4 + t^2$ is the distance from the point $\big(x(t), y(t)\big)$ on the curve to the x–axis.

3. The definite integral which gives the surface area of the surface of revolution obtained be rotating the parameterized curve around the y-axis is

$$S = \int_0^5 2\pi(t^3+4t+1)\sqrt{(3t^2+4)^2 + (4t^3+2t)^2}\ dt$$

since t^3+4t+1 is the distance from the point $\big(x(t), y(t)\big)$ on the curve to the y–axis.

Math. 126 Quiz #6 March 5, 2002

Solve the initial value problem

$$x\frac{dy}{dx} = x^2\cos x + y$$
$$y(\pi) = 1$$

. . . .

First, bring the equation to the canonical form:

$$(*) \qquad \frac{dy}{dx} + \left(-\frac{1}{x}\right)y = x\cos x$$

Next compute the integrating factor:

$$I(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}$$

Multiply the equation $(*)$ by the integrating factor, $I(x)$, to obtain:

$$\left(\frac{1}{x}y\right)' = \cos x$$

Integrate this:

$$\frac{1}{x}y = \int \cos x\,dx = \sin x + C$$

and thus

$$y = x(\sin x + C)$$

To determine C, compute $y(\pi)$ two ways:

$$1 = y(\pi) = \pi(\sin(\pi) + C) = \pi C$$

so $C = \frac{1}{\pi}$ and the solution is:

$$y = x\left(\sin x + \frac{1}{\pi}\right)$$

Math. 126 Quiz #5 February 26, 2002

Which improper integrals below converge and which diverge? Indicate your reasoning and be careful.

a) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x^3}}\,dx$

b) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

. . . .

(a)
$$\begin{aligned}\int_0^\infty \frac{1}{\sqrt{x^3}}\,dx &= \int_0^1 \frac{1}{\sqrt{x^3}}\,dx + \int_1^\infty \frac{1}{\sqrt{x^3}}\,dx\\ &= \lim_{t\to 0+}\int_t^1 \frac{1}{\sqrt{x^3}}\,dx + \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x^3}}\,dx\\ &= \lim_{t\to 0+}\frac{-2}{\sqrt{x}}\Bigg|_t^1 + \lim_{t\to\infty}\frac{-2}{\sqrt{x}}\Bigg|_1^t\\ &= -2 + \lim_{t\to 0+}\frac{2}{\sqrt{t}} + \lim_{t\to\infty}\frac{-2}{\sqrt{t}} + 2\\ &= +\infty\end{aligned}$$

so $\displaystyle\int_0^\infty \frac{1}{\sqrt{x^3}}\,dx$ diverges.

b) Use the Comparison Theorem. Note $0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$ since $-1 \le \sin x \le 1$. Evaluate $\displaystyle\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty}\frac{-1}{x}\Bigg|_1^t = \lim_{t\to\infty}\frac{-1}{t} + 1 = 1 + 0$. Since $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ converges and since $0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$, $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges.

Math. 126 Quiz #4 February 19, 2002

Expand
$$\frac{2x^3+5x^2+5x+5}{(x+1)^2(x^2+2)}$$
using the method of partial fractions.
Remark: Indicate your setup clearly since half the points are for the correct setup.

. . . .

Setup:
$$\frac{2x^3+5x^2+5x+5}{(x+1)^2(x^2+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2}$$

Solve: Clear denominators by multiplying through by $(x+1)^2(x^2+2)$:

$$2x^3+5x^2+5x+5 = A(x+1)(x^2+2) + B(x^2+2) + (Cx+D)(x+1)^2 \tag{1}$$
$$= A(x^3+x^2+2x+2) + B(x^2+2) + C(x^3+2x^2+x) + D(x^2+2x+1) \tag{2}$$

Set $x = -1$ in (1). This give $3 = 3B$ so $B = 1$.
Setting $B = 1$ and collecting like terms in (2) gives the system of equations

$$\begin{aligned} 2 &= A + C & (3) \\ 5 &= A + 2C + D + 1 & \\ 5 &= 2A + C + 2D & (4) \\ 5 &= 2A + D + 2 & (5) \end{aligned}$$

Solve: $C = 2 - A$ from (3); $D = 3 - 2A$ from (5); so (4) becomes

$$5 = 2A + (2 - A) + 2(3 - 2A) \ ;$$

so $3A = 3$; $A = 1$. Then $C = 2 - A = 1$ and $D = 3 - 2A = 1$. Hence

$$\frac{2x^3 + 5x^2 + 5x + 5}{(x+1)^2(x^2+2)} = \frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{x+1}{x^2+2}$$

Math. 126 Quiz #3 February 12, 2002

Evaluate the integral

$$\int \sec^2(\sqrt{x}\,)\, dx \ .$$

Hint: First do a substitution and then an integration by parts.

. . . .

First, apply the substitution $t = \sqrt{x}$, so $x = t^2$ and $dx = 2t\, dt$
(or $dt = \frac{dx}{2\sqrt{x}}$, $dx = 2\sqrt{x}\, dt = 2t\, dt$).

Then $\int \sec^2(\sqrt{x}\,)\, dx = \int \sec^2 t \cdot 2t\, dt = 2\int t \sec^2 t\, dt$.

To compute $\int t \sec^2 t\, dt$ use integration by parts with

$$\begin{aligned} dv &= \sec^2 t\, dt & v &= \tan t \\ u &= t & du &= dt \end{aligned}$$

so $\int t \sec^2 t\, dt \ = \ t \tan t - \int \tan t\, dt = t \tan t - \ln|\sec t| + C$.

Hence $\int \sec^2(\sqrt{x}\,)\, dx = 2\Big(\sqrt{x}\ \tan(\sqrt{x}\,) \ - \ \ln\big|\sec(\sqrt{x}\,)\big|\Big) + C$.

Remark: Some people computed $\int \tan t$ by substitution instead of from memory.
$\int \tan t\, dt = \int \frac{\sin t}{\cos t}\, dt = -\int \frac{du}{u} = -\ln|\cos t| + C = \ln|\sec t| + C$ after the substitution $u = \cos t$, $du = -\sin t\, dt$.

Math. 126 Quiz #2 January 29, 2002

1. Solve the equation $3^{2x} = 5$ for x. An answer involving ln of other numbers is fine.

2. Use logarithmic differentiation to find $\frac{dy}{dx}$ if

$$y = \frac{(x^2-1)^{2.3}(x^3+2)^{1.1}}{(x^2+1)^{0.4}} \ .$$

. . . .

1. $\ln\left(3^{2x}\right) = \ln 5$
   $2x \ln 3 = \ln 5$
   $2x = \frac{\ln 5}{\ln 3}$
   $2x = \frac{\ln 5}{2\ln 3}$

2. $\ln y = \ln\left(\frac{(x^2-1)^{2.3}(x^3+2)^{1.1}}{(x^2+1)^{0.4}}\right) = \ln\left((x^2-1)^{2.3}\right) + \ln\left((x^3+2)^{1.1}\right) - \ln\left((x^2+1)^{0.4}\right)$
   $\ln y = 2.3\ \ln(x^2-1) + 1.1\ \ln(x^3+2) - 0.4\ \ln(x^2+1)$

Differentiating both sides,

$\frac{d}{dx}\left(\ln y\right) = \frac{d}{dx}\left(2.3\ \ln(x^2-1) + 1.1\ \ln(x^3+2) - 0.4\ \ln(x^2+1)\right)$ or

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2.3(2x)}{x^2-1} + \dfrac{1.1(3x^2)}{x^3+2} - \dfrac{0.4(2x)}{x^2+1} = \dfrac{(4.6)x}{x^2-1} + \dfrac{(3.3)x^2}{x^3+2} - \dfrac{(0.8)x}{x^2+1}$ So

$$\frac{dy}{dx} = \left(\frac{(x^2-1)^{2.3}(x^3+2)^{1.1}}{(x^2+1)^{0.4}}\right)\left(\frac{(4.6)x}{x^2-1} + \frac{(3.3)x^2}{x^3+2} - \frac{(0.8)x}{x^2+1}\right)$$

Math. 126 Quiz #1 January 22, 2002

The function, $f(x) = x^3 - 3x^2 + x$ for x in the interval $[-2, 0]$, has an inverse function because f is strictly increasing on this interval.

a. What is the domain of the inverse function, f^{-1}?
b. What is the value of the derivative of f^{-1} for $x = -5$?

. . . .

a. Domain (f^{-1}) = Range(f). Since f is increasing on $[-2, 0]$ the range of f is $\left[f(-2), f(0)\right]$ which is $[-22, 0]$.
b. First note that $f^{-1}(5) = y$ if and only if $-5 = f(y)$, or $-5 = x^3 - 3x^2 + x$ or $0 = x^3 - 3x^2 + x + 5$. Since $-5 \in [-22, 0]$ there is precisely one solution to this equation in the interval $[-2, 0]$ and by trial and error you find that $f(-1) = -5$. In other words, $f^{-1}(-5) = -1$.
   Next compute $f'(x) = 3x^2 - 6x + 1$.
   Then $\left(f^{-1}\right)'(-5) = \dfrac{1}{f'\left(f^{-1}(-5)\right)} = \dfrac{1}{f'(-1)} = \dfrac{1}{10}$.
