Exam I
February 3, 2004

12. Use logarithmic differentiation to find $f'(x)$ if

$$f(x) = \frac{x\ \sin^2(x+7)}{(2x^3+2)^8}\ .$$

$$\ln\bigl(f(x)\bigr) = \ln(x) + 2\ln\bigl(\sin(x+7)\bigr) - 8\ln(2x^3+2)$$

so

$$\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{2\cos(x+7)}{\sin(x+7)} - \frac{8(6x^2)}{2x^3+2}$$

and

$$f'(x) = \left(\frac{x\ \sin^2(x+7)}{(2x^3+2)^8}\right)\left(\frac{1}{x} + \frac{2\cos(x+7)}{\sin(x+7)} - \frac{48x^2}{2x^3+2}\right)$$

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13. Evaluate the following limit.

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x}\ .$$

Use l'Hospital's Rule twice:

$$\begin{aligned}
\lim_{x\to\infty} \frac{(\ln x)^2}{x} &= \lim_{x\to\infty} \frac{2(\ln x)(1/x)}{1} && \text{[l'Hospital for "}\infty/\infty\text{"]} \\
&= \lim_{x\to\infty} \frac{2\ln x}{x} \\
&= \lim_{x\to\infty} \frac{2(1/x)}{1} && \text{[l'Hospital for "}\infty/\infty\text{"]} \\
&= \lim_{x\to\infty} \frac{2}{x} \\
&= 0\ .
\end{aligned}$$

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14. To evaluate $\int x \ln x\, dx$ apply integration by parts. There are several ways to proceed.

Choice 1: $dv = dx$. Then $u = x \ln x$; $v = x$ and $du = \big(\ln x + x\frac{1}{x}\big)dx = \big(\ln x + 1\big)dx$, so $\int x \ln x\, dx = x^2 \ln x - \int x\big(\ln x + 1\big)dx = x^2 \ln x - \int x \ln x dx - \int x\, dx = x^2 \ln x - \int x \ln x dx - \frac{1}{2}x^2$. Now solve for the integral: $2\int x \ln x\, dx = x^2 \ln x - \frac{1}{2}x^2 + C$, so

$$\int x \ln x\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

Choice 2: $dv = x\, dx$. Then $u = \ln x$; $v = \frac{1}{2}x^2$ and $du = \frac{1}{x}\, dx$, so $\int x \ln x\, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \frac{1}{x}\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$.

Choice 3: $dv = \ln x\, dx$. Then $u = x$; $v = x \ln x - x$ and $du = dx$, so $\int x \ln x\, dx = x^2 \ln x - x^2 - \int \big(x \ln x - x\big)dx = x^2 \ln x - x^2 - \int x \ln x\, dx - \int x\, dx = x^2 \ln x - x^2 + \int x \ln x\, dx + \frac{1}{2}x^2 = x^2 \ln x - \frac{1}{2}x^2 + \int x \ln x\, dx$. Solve for the integral as in Choice 1.