Exam II
March 16, 2004

12. Evaluate:

$$\int \frac{x^2}{\sqrt{1-4x^2}}\,dx.$$

Try a trig. substitution with $\sin\theta = 2x$. Then $dx = \frac{1}{2}\cos\theta\,d\theta$, $x = \frac{\sin\theta}{2}$ and $\sqrt{1-4x^2} = \cos\theta$, so

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-4x^2}}\,dx &= \int \frac{\sin^2\theta}{4\cos\theta}\frac{1}{2}\cos\theta\,d\theta = \frac{1}{8}\int \sin^2\theta\,d\theta = \frac{1}{8}\int \frac{1-\cos 2\theta}{2}\,d\theta \\ &= \frac{1}{16}\left(\theta - \frac{\sin(2\theta)}{2}\right) + C = \frac{1}{16}\big(\theta - (\sin\theta)(\cos\theta)\big) + C\end{aligned}$$

Now $\theta = \arcsin(2x)$, $\sin\theta = 2x$ and $\cos\theta = \sqrt{1-4x^2}$, so

$$\int \frac{x^2}{\sqrt{1-4x^2}}\,dx = \frac{1}{16}\left(\arcsin(2x) - 2x\sqrt{1-4x^2}\right) + C$$

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13. Consider the integral

$$\int_0^2 (2x+3)\,dx.$$

(a) (5 pts.) Evaluate this integral exactly.
(b) (8 pts.) Using the Trapezoidal Rule with $n = 4$ find an approximation to the integral.
(c) (2 pts.) Explain your answer in part (b).**Hint:**Consider the error.

(a) $\int_0^2 (2x+3)\,dx = x^2 + 3x\Big|_0^2 = (2^2 + 3\cdot 2) - (0^2 + 3\cdot 0) = 10.$
(b)

$$\begin{aligned}T_4 &= \frac{h}{2}\left[f(0) + 2f(1/2) + 2f(1) + 2f(3/2) + f(2)\right] \\ &= \frac{1/2}{2}\left[3 + 2(4) + 2(5) + 2(6) + 7\right] = \frac{40}{4} = 10.\end{aligned}$$

(c) The error bound for the Trapezoidal Rule involves the *second* derivative of the integrand, $f(x)$. Notice that for this problem $f''(x) = 0$ so that we may take $K = 0$ and

$$|E_T| \le \frac{K(b-a)^3}{12n^2} = 0.$$

The error is guaranteed to be zero.

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14. Find the area of the surface obtained by rotating the curve about the y–axis:

$$x = \sqrt{a^2 - y^2} \qquad 0 \leq y \leq a.$$

Specifically,

(a) (9 pts.) Write an integral which gives the surface area using the general formula for surface area.

(b) (6 pts.) Evaluate your integral.

Solution 1: For revolution about the y-axis

$$A = \int 2\pi x ds = \int_0^a 2\pi f(y)\sqrt{1 + \left(f'(y)\right)^2}\, dy.$$

We have

$$f'(y) = \frac{1}{2}(a^2 - y^2)^{-1/2}(-2y) = \frac{-y}{\sqrt{a^2 - y^2}}.$$

Thus

$$\begin{aligned} A &= \int_0^a 2\pi\sqrt{a^2 - y^2}\sqrt{1 + \left(\frac{-y}{\sqrt{a^2 - y^2}}\right)^2}\, dy \\ &= \int_0^a 2\pi\sqrt{a^2 - y^2}\sqrt{1 + \frac{y^2}{a^2 - y^2}}\, dy = \int_0^a 2\pi\sqrt{a^2 - y^2 + y^2}\, dy \\ &= \int_0^a 2\pi\sqrt{a^2}\, dy = \int_0^a 2\pi a dy = 2\pi a y\Big|_0^a = 2\pi a(a - 0) \\ &= 2\pi a^2. \end{aligned}$$

Solution 2:

$$A = \int 2\pi x ds = \int_0^a 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dx.$$

We have $x = \sqrt{a^2 - y^2}$ for $0 \leq y \leq a$ implies $y = \sqrt{a^2 - x^2}$ for $0 \leq x \leq a$, and

$$\frac{dx}{dy} = \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{a^2 - x^2}}.$$

Thus

$$\begin{aligned} A &= \int_0^a 2\pi x\sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2}\, dx = \int_0^a 2\pi x\sqrt{1 + \frac{x^2}{a^2 - x^2}}\, dx \\ &= \int_0^a 2\pi x\sqrt{\frac{a^2 - x^2 + x^2}{a^2 - x^2}}\, dx = \int_0^a 2\pi x\frac{a}{\sqrt{a^2 - x^2}}\, dx = -\int_{a^2}^0 \pi a\frac{1}{\sqrt{u}}\, du \end{aligned}$$

for $u = a^2 - x^2$ and thus $du = -2xdx$. Thus

$$\begin{aligned} A &= -\pi a 2\sqrt{u}\Big|_{a^2}^{0}{}^{2} = -2\pi a(0 - a) \\ &= 2\pi a^2. \end{aligned}$$