12.

A. Show that the function $f(x) = 2x + \sin(x)$ has an inverse function on the interval $(-\infty, \infty)$.

B. Compute the derivative of the inverse function at 2π.

A. f will have an inverse on $(-\infty, \infty)$ provided it is either increasing to else decreasing across the entire interval. To check this, calculate $f'(x) = 2 + \cos(x) \geq 2 + (-1) = 1 > 0$ so f is increasing.
Remarks: Here is an example where you can not write down a formula for the inverse function. It is also true that f will have an inverse if f is one–to–one, but you have no hope of showing this. In particular, it is not enough to check that two values you select go to two different values when you plug them in.

B. If g denotes the inverse function, the relevant formula is

$$g'\big(f(x)\big) = \frac{1}{f'(x)} \ .$$

Since you have been asked to calculate $g'(2\pi)$, the first step is to solve $f(x) = 2\pi$ or $2x + \sin(x) = 2\pi$. Either guess $x = \pi$ is a solution (which is easy to check once you guess it), or proceed as follows. I don't understand the $\sin x$ part of the equation very well, but it isn't very big so the solution ought to be somewhere near the solution to $2x = 2\pi$, or $x = \pi$. How far off are we? Not at all. Plugging in

$$g'(2\pi) = g'\big(f(\pi)\big) = \frac{1}{f'(\pi)} = \frac{1}{2 + \cos(\pi)} = \frac{1}{2 + (-1)} = \frac{1}{1} = 1 \ .$$

13. Evaluate $\int \frac{e^{\frac{1}{x}}}{x^3}\ dx$.

Use the following substitution:

$$u = \frac{1}{x}, \quad du = -\frac{1}{x^2}dx\ .$$

This results in the integral:

$$\int \frac{e^{\frac{1}{x}}}{x^3}\ dx = -\int e^u u\ du$$

which can be integrated by parts as follows

$$\begin{array}{cc} dy = e^u du & y = e^u \\ w = u & dw = du \end{array}$$

$$\int \frac{e^{\frac{1}{x}}}{x^3}\ dx = -\int e^u u\ du = -\left(ue^u - \int e^u du\right) = -ue^u + e^u + C = -\frac{e^{\frac{1}{x}}}{x} + e^{\frac{1}{x}} + C\ .$$

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14. Evaluate $\lim_{x\to 0} \frac{\ln(1+x) - x}{x \sin x}$.

Since the numerator and denominator each have limit zero, L'Hospital's Rule can be applied to get

$$\lim_{x\to 0} \frac{\frac{1}{1+x} - 1}{\sin x + x\cos x}$$

Again the numerator and denominator each have limit zero, so L'Hospital's Rule can be applied again to get

$$\lim_{x\to 0} \frac{-\frac{1}{(1+x)^2}}{\cos x + \cos x - x\sin x} = -\frac{1}{2}\ .$$
