

Exam II  
March 16, 2004

12. Evaluate:

$$\int \frac{x^2}{\sqrt{1-4x^2}} dx.$$

Try a trig. substitution with  $\sin \theta = 2x$ . Then  $dx = \frac{1}{2} \cos \theta d\theta$ ,  $x = \frac{\sin \theta}{2}$  and  $\sqrt{1-4x^2} = \cos \theta$ , so

$$\begin{aligned} \int \frac{x^2}{\sqrt{1-4x^2}} dx &= \int \frac{\sin^2 \theta}{4 \cos \theta} \frac{1}{2} \cos \theta d\theta = \frac{1}{8} \int \sin^2 \theta d\theta = \frac{1}{8} \int \frac{1 - \cos 2\theta}{2} d\theta \\ &= \frac{1}{16} \left( \theta - \frac{\sin(2\theta)}{2} \right) + C = \frac{1}{16} (\theta - (\sin \theta)(\cos \theta)) + C \end{aligned}$$

Now  $\theta = \arcsin(2x)$ ,  $\sin \theta = 2x$  and  $\cos \theta = \sqrt{1-4x^2}$ , so

$$\int \frac{x^2}{\sqrt{1-4x^2}} dx = \frac{1}{16} \left( \arcsin(2x) - 2x\sqrt{1-4x^2} \right) + C$$

---

13. Consider the integral

$$\int_0^2 (2x+3) dx.$$

- (a) (5 pts.) Evaluate this integral exactly.
- (b) (8 pts.) Using the Trapezoidal Rule with  $n = 4$  find an approximation to the integral.
- (c) (2 pts.) Explain your answer in part (b). **Hint:** Consider the error.

(a)  $\int_0^2 (2x+3) dx = x^2 + 3x \Big|_0^2 = (2^2 + 3 \cdot 2) - (0^2 + 3 \cdot 0) = 10.$

(b)

$$\begin{aligned} T_4 &= \frac{h}{2} [f(0) + 2f(1/2) + 2f(1) + 2f(3/2) + f(2)] \\ &= \frac{1/2}{2} [3 + 2(4) + 2(5) + 2(6) + 7] = \frac{40}{4} = 10. \end{aligned}$$

- (c) The error bound for the Trapezoidal Rule involves the *second* derivative of the integrand,  $f(x)$ . Notice that for this problem  $f''(x) = 0$  so that we may take  $K = 0$  and

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = 0.$$

The error is guaranteed to be zero.

---

14. Find the area of the surface obtained by rotating the curve about the  $y$ -axis:

$$x = \sqrt{a^2 - y^2} \quad 0 \leq y \leq a.$$

Specifically,

- (a) (9 pts.) Write an integral which gives the surface area using the general formula for surface area.  
 (b) (6 pts.) Evaluate your integral.

**Solution 1:** For revolution about the  $y$ -axis

$$A = \int 2\pi x ds = \int_0^a 2\pi f(y) \sqrt{1 + (f'(y))^2} dy.$$

We have

$$f'(y) = \frac{1}{2}(a^2 - y^2)^{-1/2}(-2y) = \frac{-y}{\sqrt{a^2 - y^2}}.$$

Thus

$$\begin{aligned} A &= \int_0^a 2\pi \sqrt{a^2 - y^2} \sqrt{1 + \left(\frac{-y}{\sqrt{a^2 - y^2}}\right)^2} dy \\ &= \int_0^a 2\pi \sqrt{a^2 - y^2} \sqrt{1 + \frac{y^2}{a^2 - y^2}} dy = \int_0^a 2\pi \sqrt{a^2 - y^2 + y^2} dy \\ &= \int_0^a 2\pi \sqrt{a^2} dy = \int_0^a 2\pi a dy = 2\pi a y \Big|_0^a = 2\pi a(a - 0) \\ &= 2\pi a^2. \end{aligned}$$

**Solution 2:**

$$A = \int 2\pi x ds = \int_0^a 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dx.$$

We have  $x = \sqrt{a^2 - y^2}$  for  $0 \leq y \leq a$  implies  $y = \sqrt{a^2 - x^2}$  for  $0 \leq x \leq a$ , and

$$\frac{dx}{dy} = \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{a^2 - x^2}}.$$

Thus

$$\begin{aligned} A &= \int_0^a 2\pi x \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2} dx = \int_0^a 2\pi x \sqrt{1 + \frac{x^2}{a^2 - x^2}} dx \\ &= \int_0^a 2\pi x \sqrt{\frac{a^2 - x^2 + x^2}{a^2 - x^2}} dx = \int_0^a 2\pi x \frac{a}{\sqrt{a^2 - x^2}} dx = - \int_{a^2}^0 \pi a \frac{1}{\sqrt{u}} du \end{aligned}$$

for  $u = a^2 - x^2$  and thus  $du = -2x dx$ . Thus

$$\begin{aligned} A &= -\pi a 2\sqrt{u} \Big|_{a^2}^0 = -2\pi a(0 - a) \\ &= 2\pi a^2. \end{aligned}$$