Exam III
April 20, 2004

12. Use the Integral Test to show that the series $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$ diverges.
Remark: Be sure to check that the Integral Test can be applied.
Solution: Seven points were allocated for the correct verification of the hypotheses in the statement of the Integral Test; eight points were given for successfully applying the theorem.

The appropriate function to use in the Integral Test is clearly $f(x) = \ln(x)/x$ since $f(n) = a_n$ for integer n. The hypotheses for using this test are:

(1 point) **Continuity**: $\ln(x)$ and $1/x$ are well–known to be continuous for all $x > 0$. Clearly their product is continuous on $[3, \infty)$.

(1 point) **Positivity**: $\ln(x)$ is positive for $x > 1$, while $1/x$ is positive for $x > 0$. Clearly their product is positive for $[3, \infty)$.

(5 points) **Decreasing**: The derivative of f is

$$f'(x) = \frac{(1/x)x - \ln(x)1}{x^2} = \frac{1 - \ln(x)}{x^2}.$$

Since the denominator is always positive for $x \neq 0$, f' is negative where $1 - \ln(x) < 0$, i.e. for $x > e$. Since $e < 3$, f' is negative on $[3, \infty)$ and f is decreasing for this interval.

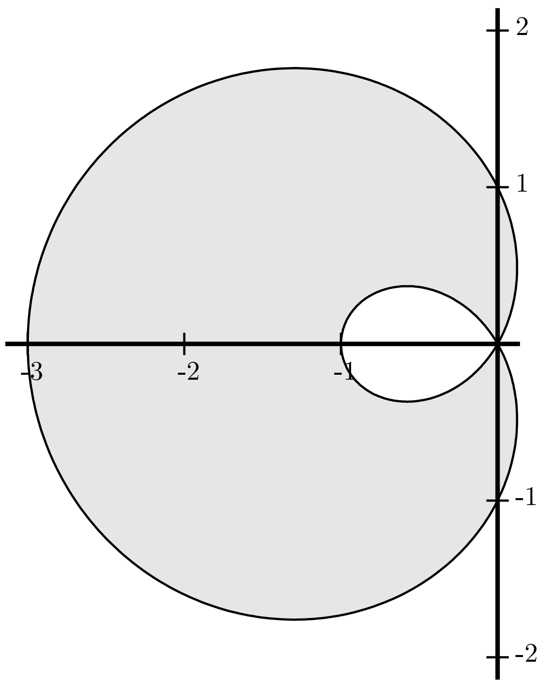
The Integral Test states that if $\int_3^{\infty} f(x)\ dx$ diverges then so does $\sum_{n=3}^{\infty} f(n)$, which is $\sum_{n=3}^{\infty} a_n$. If we let $u = \ln(x)$ then $du = (1/x)dx$ and

$$\int_3^{\infty} \frac{\ln(x)}{x}\ dx = \int u\ du = \frac{1}{2}(\ln(x))^2 \Big|_{x=3}^{x=\infty}$$

which diverges since $\ln(x) \to \infty$ as $x \to \infty$.

13. Set up an integral which computes the shaded area. The polar equation of the region is $r = 1 - 2\cos\theta$.

Remarks: $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. The answer is the difference of two areas.



This was a complicated problem to grade because there are so many correct answers. Many people computed the shaded area above the x–axis and doubled it, which got full marks if done correctly. Others computed the area directly. The crux of the problem is to work out intervals for θ which sweep out the various pieces.

From the first remark, $\theta = \frac{\pi}{3}$ is one of the angles where the curve goes through the pole (the origin). As θ increases, it gets to $\frac{\pi}{2}$ which turns out to be the point $(0, 1)$ in Cartesian coordinates. When $\theta = \pi$ we are at the point $(-3, 0)$. By the time $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$, we have traced the outside curve once and hence the integral $\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos\theta)^2 \, d\theta$ computes the total area *inside* the outer curve.

As θ continues to increase past $\frac{5\pi}{3}$, we begin to trace the inside loop. At $\theta = 2\pi$ we are at $(-1, 0)$ and by $\theta = 2\pi + \frac{\pi}{3} = \frac{7\pi}{3}$ we have traced out the entire inner loop so one answer is

$$\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos\theta)^2 \, d\theta - \frac{1}{2}\int_{\frac{5\pi}{3}}^{\frac{7\pi}{3}} (1 - 2\cos\theta)^2 \, d\theta$$

Because the curve is periodic with period 2π the limits cab be changed: a popular choice was to write the area of the inner loop as $\frac{1}{2}\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} (1 - 2\cos\theta)^2 \, d\theta$.

Another approach is to start with $\frac{1}{2}\int_0^{2\pi} (1 - 2\cos\theta)^2 \, d\theta$. This is the area inside the outer curve plus the area inside the inner curve so

$$\frac{1}{2}\int_0^{2\pi} (1 - 2\cos\theta)^2 \, d\theta - 2 \cdot \frac{1}{2}\int_{\frac{5\pi}{3}}^{\frac{7\pi}{3}} (1 - 2\cos\theta)^2 \, d\theta$$

also computes the requested area.

Computing the area above the x–axis works out as follows. From $\frac{\pi}{3}$ to π the top half of the outer curve is traced: from 0 to $\frac{\pi}{3}$ the bottom loop of the inner curve is traced. Hence another answer is

$$2\left(\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}(1-2\cos\theta)^2\,d\theta-\frac{1}{2}\int_{0}^{\frac{\pi}{3}}(1-2\cos\theta)^2\,d\theta\right)$$

One can also do

$$2\left(\frac{1}{2}\int_{0}^{\pi}(1-2\cos\theta)^2\,d\theta-2\cdot\frac{1}{2}\int_{0}^{\frac{\pi}{3}}(1-2\cos\theta)^2\,d\theta\right)$$

14. For revolution about the y-axis, surface area is

$$A=\int 2\pi x ds=\int_0^{2\pi}2\pi x(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt\ .$$

$\frac{dx}{dt}=\frac{d}{dt}(5+4\cos t)=-4\sin t$ and $\frac{dy}{dt}=\frac{d}{dt}(3\sin t)=3\cos t$. Thus

$$\begin{aligned}A&=\int_0^{2\pi}2\pi(5+4\cos t)\sqrt{(-4\sin t)^2+(3\cos t)^2}\,dt\\&=\int_0^{2\pi}2\pi(5+4\cos t)\sqrt{16\sin^2 t+9\cos^2 t}\,dt.\end{aligned}$$
