Math. 126 Quiz #1 January 21, 2003

1. Using your technique from Chapter 4, show that the function
$$f(x) = x^3 - 3x^2 - 9x + 1$$
is decreasing on the interval $[\,-1, 3\,]$.
2. Let $g(x)$ be the inverse function for $f(x)$, x in $[\,-1, 3\,]$. The equation $f(x) = 1$ has solutions $x = 0$ and $x = \frac{3\pm 3\sqrt{5}}{2}$. Use this information to find $g'(1)$.
3. Write an equation for the tangent line to the graph of $y = g(x)$ at the point $x = 1$.

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1. Compute $f'(x) = 3x^2 - 6x - 9$. $f'(x) = 0$ if and only if $3(x^2 - 2x - 3) = 3(x-3)(x+1) = 0$ if and only if $x = 3$ or $x = -1$. The function f' is continuous everywhere, so $f' < 0$ on $(-1, 3)$. Hence f is decreasing on $[-1, 3]$.
2. Since $f(x) = 1$ has solutions, $x = 0$ and $x = \frac{3\pm 3\sqrt{5}}{2}$, $g(1)$ is either 0 or one of $\frac{3\pm 3\sqrt{5}}{2}$. Since 0 is in the interval $[-1, 3]$, $g(1) = 0$. (**Note:** You weren't asked but the theory also guarantees that neither $\frac{3\pm 3\sqrt{5}}{2}$ is in the interval $[-1, 3]$.)

   The basic formula says
$$g'(1) = \frac{1}{f'(0)}$$
   and $f'(0) = 3 \cdot 0^2 - 6 \cdot 0 - 9 = -9$, so $g'(1) = \frac{1}{-9} = -\frac{1}{9}$.
3. The tangent line to the graph of $g(x)$ and $x = 1$ has slope $-\frac{1}{9}$ and goes through the point $\big(1, g(1)\big) = (1, 0)$, so an equation for it is $y - 0 = -\frac{1}{9}(x - 1)$.

Math. 126 Quiz #2 January 28, 2003

1. Solve the equation $8^{x^2} = 9$ for x. An answer such as $\ln(6) - \sqrt{\ln 8}$ or whatever is fine.
2. Find the derivative with respect to x of $(2x)^{3x}$.

. . . .

1. Take ln of both sides: $\ln\big(8^{x^2}\big) = \ln(9)$. Simplify the left hand side: $x^2 \cdot \ln(8) = \ln(9)$, or $x^2 = \frac{\ln(9)}{\ln(8)}$ or $x = \pm\sqrt{\frac{\ln(9)}{\ln(8)}}$.
2. Easiest is to let $y = (2x)^{3x}$ and use logarithmic differentiation: $\ln y = \ln\big((2x)^{3x}\big) = (3x)\ln(2x)$. Hence $\frac{y'}{y} = (3)\ln(2x) + (3x)\big(\frac{2}{2x}\big) = 3\ln(2x) + 3$. Hence $y' = y\big(3\ln(2x) + 3\big) = (2x)^{3x}\big(3\ln(2x) + 3\big)$.

   Another way:
   $(2x)^{3x} = e^{3x\ln(2x)}$, so $\frac{d\ (2x)^{3x}}{dx} = e^{3x\ln(2x)}\ \frac{d3x\ln(2x)}{dx}$, or $\frac{d\ (2x)^{3x}}{dx} = e^{3x\ln(2x)}\big((3)\ln(2x) + (3x)\frac{2}{2x}\big) = (2x)^{3x}\big(3\ln(2x) + 3\big)$.

Math. 126 Quiz #3 February 11, 2003

A charged rod of length L beginning at the origin and lying along the x–axis produces an electric field at a point (a,b) in the plane given by the integral

$$E(a,b) = \int_{-a}^{L-a} \frac{\lambda\, b}{4\pi\varepsilon_0 \left(x^2+b^2\right)^{3/2}}\, dx$$

where λ is the charge density per unit length on the rod and ε_0 is the free space permittivity. Assume that the charge density, λ, and the free space permittivity, ε_0, are constant.

Evaluate the integral to determine an expression for the electric field in terms of a, b, L, λ and ε_0.

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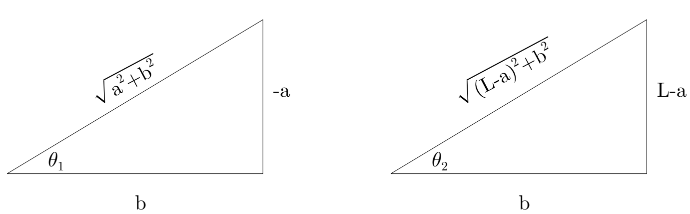
We make the substitution $x = b\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so that $x^2+b^2 = b^2\left(\tan^2\theta+1\right) = b^2\sec^2\theta$. Then $\frac{dx}{d\theta} = b\sec^2\theta$ so $dx = b\sec^2\theta d\theta$.

We want to change the limits of integration in this definite integral so, since $\theta = \arctan\left(\frac{x}{b}\right)$, let $\theta_1 = \arctan\left(\frac{-a}{b}\right)$ and $\theta_2 = \arctan\left(\frac{L-a}{b}\right)$. If you are writing this out by hand instead of cut-and-paste, let $c = \frac{\lambda}{4\pi\varepsilon_0}$. Then $E(a,b) = \displaystyle\int_{\theta_1}^{\theta_2} \frac{cb}{\left(b^2\sec^2\theta\right)^{3/2}} b\sec^2\theta\, d\theta$, or $E(a,b) = \displaystyle\int_{\theta_1}^{\theta_2} \frac{cb^2}{|b|^3}\frac{\sec^2\theta}{\sec^3\theta}\, d\theta$. Since b may be positive or negative, we must use $|b|$ where we did, but since $\theta = \arctan\left(\frac{x}{b}\right)$, it follows that θ is between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ and hence $\sec\theta > 0$, so we do not need the absolute value of $\sec\theta$.

Hence $E(a,b) = \displaystyle\frac{c}{|b|}\int_{\theta_1}^{\theta_2} \frac{1}{\sec\theta}\, d\theta = \frac{c}{|b|}\int_{\theta_1}^{\theta_2} \cos\theta\, d\theta = \frac{c}{|b|}\sin\theta\Big|_{\theta_1}^{\theta_2} = \frac{c}{|b|}\left(\sin\theta_2 - \sin\theta_1\right)$.

Analyzing the triangles with $b > 0$ shows that $\sin\theta_1 = \dfrac{-a}{\sqrt{a^2+b^2}}$ and $\sin\theta_2 = \dfrac{L-a}{\sqrt{(L-a)^2+b^2}}$. Hence

$$E(a,b) = \frac{\lambda}{4\pi\varepsilon_0\, b}\left(\frac{L-a}{\sqrt{(L-a)^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}}\right).$$

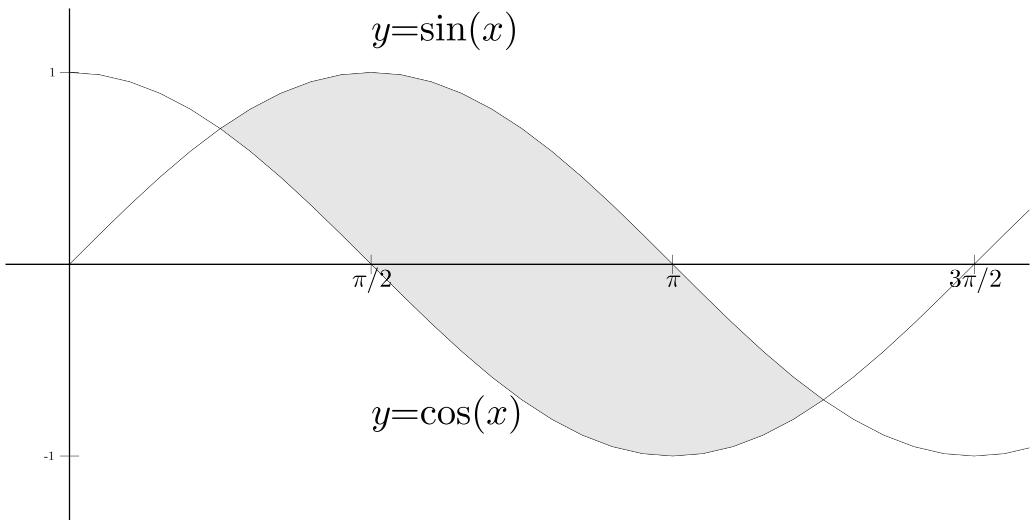


Since clearly $E(a,-b) = -E(a,b)$, the formula holds for all $b \neq 0$. For $b = 0$, clearly $E(a,0) = 0$.

Remark: If $0 \le a \le L$ and $b = 0$ then the formula is probably physically meaningless since we are actually on the rod. However, if $a < 0$ or $a > L$ and $b = 0$, then $E(a,0) = 0$ probably is physically meaningful. You can use l'Hopital's Rule to check that for $a < 0$ or $a > L$, $\lim_{b \to 0} E(a,b) = 0$.

Math. 126 Quiz #5 February 25, 2003

Find the centroid of the shaded region.



. . . .

The mass, or equivalently the area, is given by

$$A = \int_a^b (\sin x - \cos x) dx \ .$$

where a is the x–coordinate of the left–hand intersection point and b is the x–coordinate of the right–handed intersection point.

The point a satisfies $\sin(a) = \cos(a)$ or $\tan(a) = 1$ or $a = \frac{\pi}{4}$. The point b satisfies the same equation and so $b = a + \pi$ or $b = \frac{5\pi}{4}$. You could also eye–ball the points from the graph.

Hence $A = -\cos(x) - \sin(x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = -(-\frac{\sqrt{2}}{2}) - (-\frac{\sqrt{2}}{2}) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 4\frac{\sqrt{2}}{2} = 2\sqrt{2}$.

The moment about the x–axis, M_x is given by

$$M_x = \int_a^b \frac{\sin x + \cos x}{2} (\sin x - \cos x) dx$$

where a and b are as above. Now $M_x = \frac{1}{2}\displaystyle\int_a^b \sin^2(x) - \cos^2(x)\ dx = \frac{1}{2}\int_a^b \frac{-\cos(2x)}{2} dx = \frac{-1}{8}\sin(2x)\Big|_a^b = \frac{-1}{8}\sin\Big(\frac{5\pi}{2}\Big) - \frac{-1}{8}\sin\Big(\frac{\pi}{2}\Big) = \frac{-1}{8}(1) - \frac{-1}{8}(1) = 0.$

The moment about the y–axis, M_y is given by

$$M_y = \int_a^b x(\sin x - \cos x)dx$$

where a and b are as above. Integrate by parts: $dv = (\sin x - \cos x)dx$, $u = x$, so $v = -\cos x - \sin x$ and $du = dx$ so $M_y = -(\cos x + \sin x)x\Big|_a^b + \displaystyle\int_a^b \sin x + \cos x\ dx = -(\cos x + \sin x)x\Big|_a^b - \cos x + \sin x\Big|_a^b = -\cos x(x+1) + \sin x(1-x)\Big|_a^b = -2x\cos x\Big|_a^b$ since $\sin(x) = \cos(x)$ for $x = a$ and for $x = b$.

Hence $M_y = -2\Big(\frac{5\pi}{4}\Big)\Big(-\frac{\sqrt{2}}{2}\Big) + 2\Big(\frac{\pi}{4}\Big)\Big(\frac{\sqrt{2}}{2}\Big) = \frac{6\pi\sqrt{2}}{4} = \frac{3\pi\sqrt{2}}{2}.$

Therefore, the center of mass is at $(\bar{x}, \bar{y}) = \Big(\frac{M_y}{A}, \frac{M_x}{A}\Big) = \Big(\frac{\frac{3\pi\sqrt{2}}{2}}{2\sqrt{2}}, 0\Big) = \Big(\frac{3\pi}{4}, 0\Big).$

Looking at the graph, this is not unreasonable.

Math. 126 Quiz #6 March 4, 2003

Solve the initial value problem

$$x\frac{dy}{dx} = x^2 \tan x + y$$

$$y(\pi/4) = 1$$

. . . .

The equation is linear but not in standard form. The standard form is

$$\frac{dy}{dx} + \frac{-1}{x} y = x\tan x\ .$$

Hence $P(x) = \frac{1}{x}$ and $\int P dx = -\ln|x| + C$. Hence we may use $I = e^{-\ln|x|} = \frac{1}{|x|}$ as an integrating factor, and even use $I = \frac{1}{x}$.

Check: $\frac{d\frac{1}{x}y}{dx} = \frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x}\Big(y' - \frac{1}{x}y\Big)$, so $I = \frac{1}{x}$ is an integrating factor.

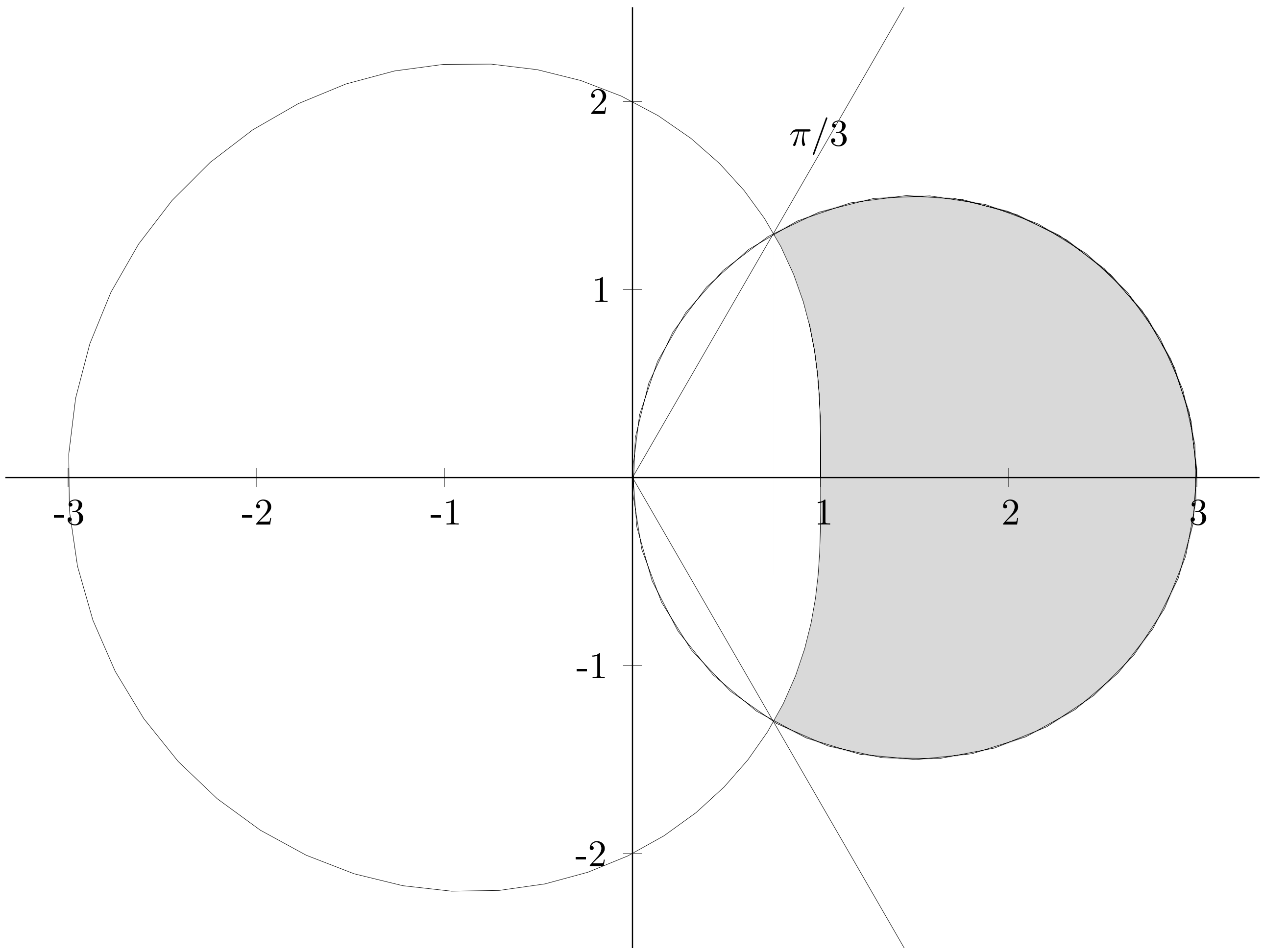
Hence $y = \frac{1}{I}\int I\cdot Q dx$, where $Q(x) = x\tan x$, and $I\cdot Q = \tan x$. Therefore $\int I\cdot Q dx = \int \tan x dx = \ln|\sec(x)| + C$ and $\frac{1}{x}y = \ln|\sec(x)| + C$, or $y = x(\ln|\sec x| + C)$.

To solve the initial value problem, note $y(\pi/4) = 1$ and $y(\pi/4) = \frac{\pi}{4}(\ln|\sec(\pi/4)| + C)$, so $1 = \frac{\pi}{4}(\ln\sqrt{2} + C)$, or $C = \frac{4}{\pi} - \ln\sqrt{2}$. Hence

$$y = x\Big(\ln|\sec x| + \frac{4}{\pi} - \ln\sqrt{2}\ \Big)\ .$$

Math. 126 Quiz #7 March 26, 2003

Find the area of the region inside the polar curve $r = 3\cos\theta$ and outside the polar curve $r = 2 - \cos\theta$.



. . . .

We have shaded the required region. We start by locating the initial and terminal angles. We *hope* that they satisfy the equation $3\cos\theta = 2 - \cos\theta$. This equation is equivalent to $4\cos\theta = 2$ or $\cos\theta = \frac{1}{2}$ and hence $\theta = \pm\frac{\pi}{3}$. We are looking for two angles and we have found two angles.

It now follows that the area is given by

$$\begin{aligned} A =& \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left((3\cos\theta)^2 - (2-\cos\theta)^2 \right)\, d\theta \\ =& \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(9\cos^2\theta - \left(4 - 4\cos\theta + \cos^2\theta\right)\right)\, d\theta \\ =& \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(8\cos^2\theta + 4\cos\theta - 4\right)\, d\theta\ . \end{aligned}$$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 1\ d\theta = \frac{2\pi}{3}\ ;$$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\theta\ d\theta = \sin\theta\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \frac{2\sqrt{3}}{2} = \sqrt{3}\ ;$$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2\theta\ d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1+\cos(2\theta)}{2}\ d\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4}\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = 2\frac{\pi}{6} + 2\frac{\frac{\sqrt{3}}{2}}{4} = \frac{\pi}{3} + \frac{\sqrt{3}}{4}\ .$$

Hence

$$A = 4\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) + 2\left(\sqrt{3}\right) - 2\left(\frac{2\pi}{3}\right) = 3\sqrt{3}\ .$$

Math. 126 Quiz #8 April 2, 2003

Consider the ellipse in the xy-plane which has foci at $(3,2)$ and $(3,8)$ with eccentricity $\frac{1}{2}$.

1. What are the coordinates of the center?
2. What are the coordinates of the vertices?
3. What is the equation of the ellipse?
4. What are the equations of the directrixes?

. . . .

The center is at $\left(\frac{3+3}{2}, \frac{2+8}{2}\right) = (3,5)$. The major axis is parallel to the y axis.
The distance from the center to a focus is $c = 8-5 = 3$ or $c = 5-2 = 3$.
Since $e = \frac{c}{a}$, $\frac{1}{2} = \frac{3}{a}$, so $a = 6$: the vertices are at $(3, 5\pm 6)$ or at $(3,11)$ and $(3,-1)$.
Since $b^2 = a^2 - c^2$ for an ellipse, $b^2 = 36 - 9 = 27$. An equation for our ellipse is

$$\frac{(x-3)^2}{27} + \frac{(y-5)^2}{36} = 1\ .$$

The distance from the vertex $(3,-1)$ to the focus $(3,2)$ is 3. If k is the distance from the vertex $(3,-1)$ to the associated directrix, then $e = \frac{3}{k}$, so $k = 6$ and so the directixes are the lines $y = -1-6 = -7$ and $y = 11+6 = 17$.

Math. 126 Quiz #9 April 8, 2003

Which of the following series converge and which diverge? Why?

A. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,!}$:

Recall $n\,! = n(n-1)\cdots 2 \cdot 1$

B. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{1+\sqrt{n}}$:

C. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{2^n}{3^n} + \frac{1}{n^2}\right)$:

D. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{\pi^n}{3^n} + \frac{1}{n^2}\right)$:

. . . .

A. By the Ratio test $\displaystyle\lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$ and $0 < 1$ so series converges.

B. $\displaystyle\lim_{n\to\infty} \frac{n}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\frac{1}{\sqrt{n}}+1} = \infty$. Since $\lim_{n\to\infty} a_n \neq 0$, the series diverges. An equally good calculation is $\displaystyle\lim_{n\to\infty} \frac{n}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{n} = \infty$.

C. $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{3^n}$ is a geometric series with $r = \frac{2}{3} < 1$ and so converges: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a p–series with $p = 2 > 1$ and so it converges. Therefore the sum of the two convergent series converges.

D. This time $\displaystyle\sum_{n=2}^{\infty} \frac{\pi^n}{3^n}$ is a geometric series with $r = \frac{\pi}{2} > 1$ and so it diverges. This suggests using the Comparison Test: $\frac{\pi^n}{2^n} < \frac{\pi^n}{3^n} + \frac{1}{n^2}$ and $\displaystyle\sum_{n=2}^{\infty} \frac{\pi^n}{3^n}$ diverges: therfore $\displaystyle\sum_{n=2}^{\infty} \left(\frac{\pi^n}{3^n} + \frac{1}{n^2}\right)$ diverges.

Math. 126 Quiz #10 April 15, 2003

Consider the function $f(x)$ defined by the power series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^{\frac{3}{2}}}$$

1. Find the interval of convergence of the power series for f.

2. Write down the power series for $f'(x)$.
3. Find the interval of convergence of the power series for f'.

. . . .

1. Via the Ratio Test: $\left|\frac{\frac{x^{n+1}}{(n+1)^{3/2}}}{\frac{x^n}{n^{3/2}}}\right| = \left|\frac{x^{n+1}n^{3/2}}{x^n(n+1)^{3/2}}\right| = |x|\frac{n^{3/2}}{(n+1)^{3/2}}$.

   In the limit, $\lim_{n\to\infty} |x|\frac{n^{3/2}}{(n+1)^{3/2}} = |x|$, since we can compute the limit of an algebraic quotient by examining the highest power of n in the numerator and the highest power of n in the denominator. It follows that the radius of convergence of this power series is 1. When $x = \pm 1$, $\sum_{n=1}^{\infty}\left|\frac{(\pm 1)^n}{n^{\frac{3}{2}}}\right|$ is a p–series with $p = \frac{3}{2} > 1$ and hence convergent. Thus the original two series, $\sum_{n=1}^{\infty}\frac{(\pm 1)^n}{n^{\frac{3}{2}}}$, are absolutely convergent, hence convergent. Therefore the domain of f is the interval $[-1, 1]$ (and the convergence is absolute in the entire interval).
2. By our theorem, $f'(x) = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n^{\frac{1}{2}}}$ or $\sum_{n=0}^{\infty}\frac{x^n}{(n+1)^{\frac{1}{2}}}$.
3. The radius of convergence for f' is the same as for f, and hence it is 1. This time the two series we need to examine are $\sum_{n=0}^{\infty}\frac{(\pm 1)^n}{(n+1)^{\frac{1}{2}}}$. When $x = +1$, we have a p–series with $p = \frac{1}{2} < 1$ so the series diverges. When $x = -1$ the series is alternating (obvious), the terms go to 0 (also obvious) and the terms are decreasing ($g(x) = x^{-1/2}$, so $g'(x) = -\frac{1}{2}x^{-3/2}$ and when $x > 0$, $x^{-3/2} > 0$ so $g'(x) < 0$ and the terms are decreasing). Hence this series converges conditionally and the domain of $f'(x)$ is $[-1, 1)$.