Math. 126 Quiz #1 January 20, 2004

1. The function
$$f(x) = x^5 + 5x^2 - 15x + 4$$
is increasing on the interval $[1,3]$ and hence one–to–one. Let $g(x)$ be the inverse function for $f(x)$, x in $[1,3]$. The equation $f(x) = 26$ has a solution, $x = 2$. Use this information to find $g'(26)$.

2. Write an equation for the tangent line to the graph of $y = g(x)$ at the point $x = 26$.

. . . .

Since $f(2) = 26$, $g(26) = 2$. Since $g'\big(f(x)\big) = \frac{1}{f'(x)}$, $g'(26) = g'\big(f(2)\big) = \frac{1}{f'(2)}$. Since $f'(x) = 5x^4 + 10x - 15$, $f'(2) = 5 \cdot 2^4 + 10 \cdot 2 - 15 = 5 \cdot 16 + 20 - 15 = 80 + 20 - 15 = 85$ and $g'(26) = \frac{1}{85}$.

Since $g(26) = 2$ and $g'(26) = \frac{1}{85}$, an equation of the tangent line is
$$y - 2 = \frac{1}{85}(x - 26) \ .$$

Math. 126 Quiz #2 January 27, 2003

1. Solve the equation $9^{3x+4} = 2$ for x. An answer such as $\ln(6) - \sqrt{\ln 8}$ or whatever is fine.

2. Find the derivative with respect to x of $(\tan x)^{3x}$.

. . . .

Since $9^{3x+4} = 2$, $\ln\big(9^{3x+4}\big) = \ln 2$, so $(3x+4)\ln 9 = \ln 2$ and so $3x + 4 = \frac{\ln 2}{\ln 9}$. Hence $3x = \frac{\ln 2}{\ln 9} - 4$ and $x = \frac{\frac{\ln 2}{\ln 9} - 4}{3}$. If you prefer, you can write this as $x = \frac{\ln 2 - 4\ln 9}{3\ln 9}$.

Write $(\tan x)^{3x} = e^{3x\ln(\tan x)} = e^u$. Since $\frac{d\,e^u}{dx} = u'e^u$, it behooves us to find u'. Well, $\frac{d\,\big(3x(\ln\tan x)\big)}{dx} = 3(\ln\tan x) + 3x\Big(\frac{\sec^2 x}{\tan x}\Big)$, so
$$\frac{d\,(\tan x)^{3x}}{dx} = \Big(3(\ln\tan x) + 3x(\frac{\sec^2 x}{\tan x})\Big)(\tan x)^{3x} \ .$$

Math. 126 Quiz #3 February 10, 2004

Evaluate the integral
$$\int \sec^2(\sqrt{x}\,)\,dx \ .$$

Hint: First do a substitution and then an integration by parts.

. . . .

The square root inside the sec does not look friendly, so let $u = \sqrt{x}$. Then $du = \frac{dx}{2\sqrt{x}}$. There is no "extra" square root, so let $dx = 2udu$ and

$$\int \sec^2(\sqrt{x}\,)\,dx = 2\int u\sec^2 u\,du\ .$$

This new integral looks good for parts. Taking dv to be the biggest chunk you can integrate suggests $dv = \sec^2 u\,du$ or $dv = udu$. This second try looks not so good since $v = \frac{u^2}{2}$ and the power of u in the next integral will be bigger while the derivative of the sec squared will still be trig functions. So let's go with $dv = \sec^2 u\,du$ and $w = u$: $v = \tan u$ and $dw = du$ so

$$\int u\sec^2 u\,du = u\tan u - \int \tan u\,du = u\tan u - \ln|\sec u| + C\ .$$

Hence

$$\int \sec^2(\sqrt{x}\,)\,dx = 2\Big(\sqrt{x}\tan(\sqrt{x}) - \ln\Big|\sec(\sqrt{x})\Big|\Big) + C$$

Math. 126 Quiz #4 February 17, 2004

1. Set up the initial form for the partial fraction for

$$\frac{x^3+7x^2-10x+1}{(x+2)^2(x-1)(x^2-1)^2(x^2+2x+3)^2}\ .$$

Hint: The initial form for $\frac{1}{(x+1)(x-1)}$ is $\frac{A}{x-1} + \frac{B}{x+1}$.

2. Integrate $\displaystyle\int \frac{2x^2+2x+1}{(x+1)(x^2+1)}\,dx$.

. . . .

The degree of the numerator is 3 and the degree of the denominator is 11 so there is no need to do a polynomial long division. The second quadratic is irreducible since $2^2 - 4\cdot 1\cdot 3 < 0$, but $x^2-1 = (x-1)(x+1)$. Hence

$$\frac{x^3+7x^2-10x+1}{(x+2)^2(x-1)(x^2-1)^2(x^2+2x+3)^2} = \frac{x^3+7x^2-10x+1}{(x+2)^2(x-1)^3(x+1)^2(x^2+2x+3)^2} =$$
$$\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3} + \frac{F}{x+1} + \frac{G}{(x+1)^2} +$$
$$\frac{Hx+I}{x^2+2x+3} + \frac{Jx+K}{(x^2+2x+3)^2}\ .$$

The order of the factors is not important, nor are the names of the variables: you could do the $x-1$ powers first, then the quadratic, then the $x+2$ powers, etc.

To do the integral, requires expanding $\frac{2x^2+2x+1}{(x+1)(x^2+1)}$ by partial fractions. The degree of the numerator is 2, that if the denominator 3, so no long division is needed, the quadratic is irreducible so

$$\frac{2x^2+2x+1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \ , \quad \text{so}$$

$$2x^2+2x+1 = A\cdot(x^2+1) + (Bx+C)(x+1) \ .$$

Plug in $x=-1$ and see $2(-1)^2+2(-1)+1 = A((-1)^1+1)+(Bx+C)\cdot 0$, or $2A=1$ or $A=\frac{1}{2}$.
Hence $2x^2+2x+1 = \frac{1}{2}\cdot(x^2+1)+(Bx+C)(x+1)$, so $4x^2+4x+2 = x^2+1+2(Bx+C)(x+1)$ so
$3x^2+4x+1 = 2(Bx+C)(x+1) = 2Bx^2+2(B+C)x+2C$. Hence $B=\frac{3}{2}$, $C=\frac{1}{2}$ and

$$\frac{2x^2+2x+1}{(x+1)(x^2+1)} = \frac{1}{2}\Big(\frac{1}{x+1} + \frac{3x+1}{x^2+1}\Big) \ .$$

To do the integral, compute $\displaystyle\int \frac{dx}{x+1} = \ln|x+1| + C$ and $\displaystyle\int \frac{(3x+1)dx}{x^2+1} = 3\int \frac{x\,dx}{x^2+1} + \int \frac{dx}{x^2+1}$. By substitution, $\displaystyle\int \frac{x\,dx}{x^2+1} = \frac{\ln(x^2+1)}{2} + C$, so

$$\int \frac{2x^2+2x+1}{(x+1)(x^2+1)}\,dx = \frac{\ln|x+1|}{2} + \frac{3\ln(x^2+1)}{4} + \frac{\arctan x}{2} + C \ .$$

Math. 126 Quiz #5 February 24, 2004

Which improper integrals below converge and which diverge? Indicate your reasoning and be careful.

a) $\displaystyle\int_1^\infty \frac{1}{\sqrt[9]{x^{11}}}\,dx$

b) $\displaystyle\int_{-\infty}^1 \frac{1}{\sqrt[9]{x^{11}}}\,dx$

c) $\displaystyle\int_1^\infty \frac{\sin^2 x}{\sqrt[9]{x^{11}}}\,dx$

. . . .

(a) Since $x^{-\frac{11}{9}}$ is continuous on $[1,\infty)$, $\displaystyle\int_1^\infty \frac{1}{\sqrt[9]{x^{11}}}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt[9]{x^{11}}}\,dx = \lim_{t\to\infty} \frac{x^{-\frac{2}{9}}}{-\frac{2}{9}}\Bigg|_1^t =$ $\displaystyle\lim_{t\to\infty} -\frac{9t^{-\frac{2}{9}}}{2} - \left(-\frac{9}{2}\right) = \frac{9}{2}$ since the limit as $t\to\infty$ of t to a negative power, $-\frac{2}{9}$, goes to 0. In particular, the integral converges.

(b) This one is a bit tricky since $x^{-\frac{11}{9}}$ is not continuous at $0 \in (-\infty, 1]$. Hence

$$\int_{-\infty}^{1} \frac{1}{\sqrt[9]{x^{11}}}\,dx = \int_{-\infty}^{-1} \frac{1}{\sqrt[9]{x^{11}}}\,dx + \int_{-1}^{0} \frac{1}{\sqrt[9]{x^{11}}}\,dx + \int_{0}^{1} \frac{1}{\sqrt[9]{x^{11}}}\,dx\ .$$

The -1 is arbitrary. You may use any negative number you like. We suspect that the analysis at $-\infty$ will look a lot like the analysis in part (a), so let's start with one of the other two integrals.

$$\int_{-1}^{0} \frac{1}{\sqrt[9]{x^{11}}}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{\sqrt[9]{x^{11}}}\,dx = \lim_{t\to 0^-} -\frac{9x^{-\frac{2}{9}}}{2}\Bigg|_{-1}^{t} = \frac{9}{2}\lim_{t\to 0^-} t^{-\frac{2}{9}} - \left(-\frac{9}{2}\right).$$

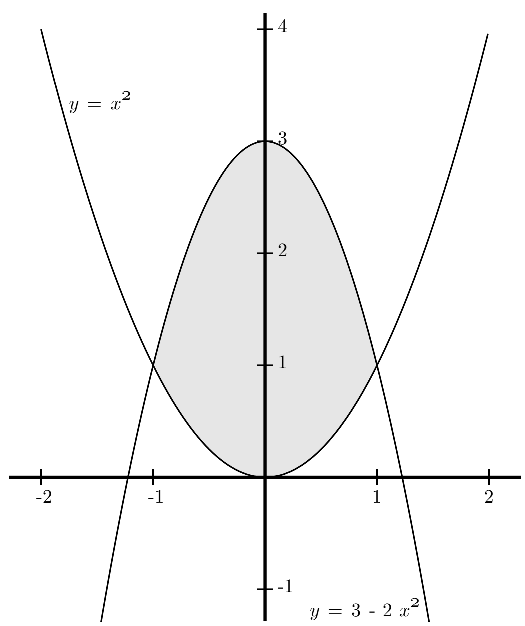
Since $-\frac{2}{9} < 0$, $\lim\limits_{t\to 0^-} t^{-\frac{2}{9}} = \infty$ and this integral diverges. Hence the original integral also diverges.

(c) You have no hope of actually doing the integral so you can not work directly with the definition. Instead, use a theorem. Since $-1 \le \sin x \le 1$, $0 \le \sin^2 x \le 1$ and hence $0 \le \frac{\sin^2 x}{\sqrt[9]{x^{11}}} \le \frac{1}{\sqrt[9]{x^{11}}}$. Since $\int_1^\infty \frac{dx}{\sqrt[9]{x^{11}}}$ converges by part (a), our theorem says $\int_1^\infty \frac{\sin^2 x\,dx}{\sqrt[9]{x^{11}}}$ also converges.

Math. 126 Quiz #6

March 2, 2004

Find the y coordinate of the centroid of the shaded region. The x coordinate is 0 by symmetry.



Find the volume of the solid of revolution obtained by rotating the shaded region about the line $x = 2$.

. . . .

To find the centroid, first find the area:

$$\int_{-1}^{1}(3-2x^2)-x^2dx = \int_{-1}^{1} 3-3x^2dx = 3x-x^3\Big|_{-1}^{1} = 3-\frac{1}{3}-(-3-\frac{-1}{3}) = \frac{5}{3}+\frac{5}{3} = \frac{10}{3}\ .$$

The moment about the x–axis is given by:

$$\begin{aligned}\int_{-1}^{1}\frac{(3-2x^2)+x^2}{2}\ \big((3-2x^2)-x^2\big)dx &= \int_{-1}^{1}\frac{(3-2x^2)^2-(x^2)^2}{2}\ dx \\ &= \frac{1}{2}\int_{-1}^{1} 9-12x^2+4x^4-x^4\ dx \\ &= \frac{1}{2}\int_{-1}^{1} 9-12x^2+3x^4\ dx = \frac{1}{2}\ (9x-4x^3+\frac{3x^5}{5})\Big|_{-1}^{1} \\ &= \frac{1}{2}\big((5+\frac{3}{5})-(-5-\frac{3}{5})\big) = 5+\frac{3}{5} = \frac{28}{5}\end{aligned}$$

The y coordinate of the center of mass is at $\frac{\frac{28}{5}}{\frac{10}{3}} = \frac{3\cdot 28}{5\cdot 10} = \frac{42}{25}$.

To do the second part, compute the distance from the center of mass at $\big(0, \frac{42}{25}\big)$ to the line $x=2$: this distance is 2. Then Pappus says that the volume is $2\pi\cdot 2\cdot\frac{10}{3} = \frac{40\pi}{3}$.

Math. 126 Quiz #7 March 23, 2004

Find the solution of the differential equation with initial value

$$y\ \frac{d\,z}{d\,y} +\ 2\ z = y^3$$

subject to the initial condition $z(1)=0$.

. . . .

This equation is linear but not in standard form: the standard form is

$$\frac{d\,z}{d\,y} +\ \frac{2}{y}\ z = y^2\ .$$

Hence $\int Pdy = \int \frac{2}{y}dy = 2\ln|y| + C$. For an integrating factor, take $e^{\int Pdy} = e^{2\ln|y|} = |y|^2$ and as usual for integrating factors one may use y^2. Hence $y^2z = \int y^2y^2dy = \frac{y^5}{5}+C$ so the general solution is

$$z = \frac{y^3}{5}+\frac{C}{y^2}$$

To get the initial value, observe

$$0 = z(1) = \frac{1^3}{5}+\frac{C}{1^2}$$

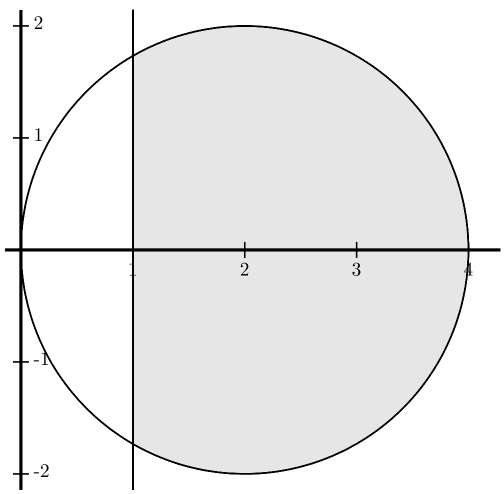
so $C = -\frac{1}{5}$ and the solution is

$$z = \frac{y^3}{5} - \frac{1}{5y^2}$$

or if you prefer $z = \frac{y^5-1}{5y^2}$.

Math. 126 Quiz #8 March 30, 2004

Find the area of the region inside the polar curve $r = 4\cos\theta$ and to the right of the line $r = \sec\theta$.



. . . .

The formula is

$$Area = \frac{1}{2} \int_{\alpha_0}^{\alpha_1} (4\cos\theta)^2 - \sec^2\theta \; d\theta$$

where α_0 and α_1 are the angles where the line and the circle intersect. These occur when $4\cos\theta = \sec\theta$, or when $4\cos^2\theta = 1$ or $\cos\theta = \pm\frac{1}{2}$. From the graph, the α_i are the angles where $\cos\theta = \frac{1}{2}$ so $\alpha_1 = \frac{\pi}{3}$ and $\alpha_0 = -\frac{\pi}{3}$.

$$\begin{aligned} Area =& \frac{1}{2} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 16\cos^2\theta - \sec^2\theta \; d\theta = \frac{1}{2} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 8 - 8\cos(2\theta) - \sec^2\theta \; d\theta \\ =& 4\,\theta + 2\sin(2\theta) - \frac{1}{2}\tan\theta \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \frac{8\pi}{3} + 2(\frac{1}{2} - \frac{-1}{2}) - \frac{1}{2}\big(\sqrt{3} - (-\sqrt{3})\big) \\ =& \frac{8\pi}{3} + 2 - \sqrt{3} \end{aligned}$$

Math. 126 Quiz #9 April 6, 2004

Tell why the following series diverges or converges. If it converges, sum it.

$$\sum_{n=0}^{\infty} \frac{2^n + 3^{2n}}{3^{3n+1}}$$

. . . .

Write $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 3^{2n}}{3^{3n+1}} = \sum_{n=0}^{\infty} \frac{2^n}{3^{3n+1}} + \sum_{n=0}^{\infty} \frac{3^{2n}}{3^{3n+1}} = \sum_{n=0}^{\infty} \frac{2^n}{3^{3n+1}} + \sum_{n=0}^{\infty} \frac{1}{3^{n+1}}$.

Each of these two series is geometric: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^{3n+1}}$ has $r = \frac{2}{27}$ and $a = \frac{1}{3}$, so $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^{3n+1}} = \frac{\frac{1}{3}}{1 - \frac{2}{27}} = \frac{\frac{1}{3}}{\frac{25}{27}} = \frac{9}{25}$ and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^{n+1}}$ has $r = \frac{1}{3}$ and $a = \frac{1}{3}$, so $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^{n+1}} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$.

Hence $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 3^{2n}}{3^{3n+1}} = \frac{9}{25} + \frac{1}{2} = \frac{18 + 25}{50} = \frac{43}{50}$.

Math. 126 Quiz #10 April 13, 2004

Tell why each of the following series diverges or converges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

. . . .

#1: This is the harmonic series, or the p–series with $p = 1$. Hence it diverges since $1 = p \leq 1$

#2: This is the p–series with $p = 2$. Hence it converges since $2 = p > 1$

#3: Since $\ln n$ grows more slowly than powers of n, *eventually* $\ln n < \sqrt{n}$ so $\dfrac{\ln n}{n^2} < \dfrac{1}{n^{\frac{3}{2}}}$.

Now $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is the p series with $\frac{3}{2} = p > 1$ so it converges. By the Comparison Test $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ also converges.

To actually check $\ln n < \sqrt{n}$ eventually, consider the function $f(x) = \sqrt{x} - \ln x$. Compute $f'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{x} = \dfrac{\sqrt{x} - 2}{2x}$ and this is positive for $x > 4$. Check

$f(4) = 2 - \ln 4$ and $e^2 > (2.5)^2 > 6 > 4$ so $\ln 4 < 2$ and $f(4) > 0$. Therefore $f(x) > 0$ for all $x \geq 4$.

Math. 126 Quiz #11 April 27, 2004

Find the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{\ln(n+1)}$$

. . . .

First compute the radius of convergence. Via the ratio test: $\left| \frac{\frac{x^{n+1}}{\ln(n+2)}}{\frac{x^n}{\ln(n+1)}} \right| = \frac{\ln(n+1)}{\ln(n+2)} |x|$.

By l'Hôpital's Rule $\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n+2)} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n+2}} = \lim_{n\to\infty} \frac{n+2}{n+1} = 1$ and hence the radius of convergence is 1.

Via the root test $\left| \sqrt[n]{\frac{x^n}{\ln(n+1)}} \right| = \frac{|x|}{\big(\ln(n+1)\big)^{\frac{1}{n}}}$. This involves computing

$$\lim_{n\to\infty} \big(\ln(n+1)\big)^{\frac{1}{n}}$$

which is an indeterminate exponential. Hence compute

$$\lim_{x\to\infty} \frac{1}{x} \ln\big(\ln(x+1)\big) = \lim_{x\to\infty} \frac{\frac{\frac{1}{x+1}}{\ln(x+1)}}{1} = \lim_{x\to\infty} \frac{1}{(x+1)\ln(x+1)} .$$

$\lim_{x\to\infty} (x+1)\ln(x+1)$ is another indeterminate form, so

$$\lim_{x\to\infty} (x+1)\ln(x+1) = \lim_{x\to\infty} \frac{\ln(x+1)}{\frac{1}{x+1}} = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{-1}{(x+1)^2}} = \lim_{x\to\infty} -(x+1) = -\infty .$$

It follows that $\lim_{x\to\infty} \frac{1}{x} \ln\big(\ln(x+1)\big) = \frac{1}{-\infty} = 0$ and hence $\lim_{n\to\infty} \big(\ln(n+1)\big)^{\frac{1}{n}} = e^0 = 1$. Again it follows that the radius of convergence is 1.

To determine the interval of convergence we need to examine the series $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$. Since $\ln(n+1)$ is eventually less than n, and since $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series which diverges, the Comparison Test shows $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ diverges.

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ is alternating and the terms go to 0: i.e. $\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$. If $f(x) = \frac{1}{\ln(x+1)}$, $f'(x) = \frac{-1}{\left(\ln(x+1)\right)^2} \frac{1}{x+1}$ which is < 0 for $x > 0$. By the Alternating Series Test $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ converges.

Hence the interval of convergence is $[-1, 1)$.
