

**Math 165: Honors Calculus I**

Name: \_\_\_\_\_

**Exam I** *Oct. 13, 1994*

There are 9 problems worth 12 points each. Problem 10 is a bonus question worth an additional 12 points.

1. a) (4 pts) Define the positive integers, .

b) (4 pts) Define the rational numbers, .

c) (4 pts) Assume  $\sqrt{3}$  is irrational. Prove that  $(\sqrt{3} - 1)^3$  is irrational.

2. a) (4 pts) Define an inductive set.

b) (4 pts) State the PRINCIPLE OF MATHEMATICAL INDUCTION.

c) (4 pts) Prove by induction that for  $n \in$

$$(x - 1) \sum_{k=0}^{n-1} x^k = x^n - 1$$

3. a) (4 pts) Define  $\binom{n}{k}$ .

b) (4 pts) State the BINOMIAL THEOREM.

c) (4 pts) Find the coefficient of  $x^{14}$  in  $(x^2 + 3)^{20}$ .

4. a) (4 pts) Define what it means for a set  $S$  of real numbers to be bounded from above.

b) (4 pts) Define  $\sup S$ .

c) (4 pts) Let  $S = \left\{ \frac{n}{n+1} \mid n \in \mathbb{N} \right\}$ . Prove that  $\sup S = 1$ .

5. Give a precise definition of the following.

a) (4 pts) A step function  $s$  on  $[a, b]$ .

b) (4 pts)  $\int_a^b s(x) dx$  where  $s$  is a step function on  $[a, b]$ .

c) (4 pts) An integrable function  $f$  on  $[a, b]$ .

6. (12 pts) Let

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \text{ and } 0 \leq x \leq 1 \\ 1 & \text{if } x \notin \mathbb{Q} \text{ and } 0 \leq x \leq 1 \end{cases}$$

a) (4 pts) Show that  $\underline{I}(f) = 0$ .

b) (4 pts) Show that  $\bar{I}(f) = 1$ .

c) (4 pts) Prove that  $f$  is *not* integrable on  $[0, 1]$ .

7. Give precise statements of the following theorems for integrals.

a) (4 pts) THE LINEARITY THEOREM.

b) (4 pts) THE COMPARISON THEOREM.

c) (4 pts) THE EXPANSION/CONTRACTION THEOREM.

8. Evaluate the following integrals (justify your answers!).

a) (4 pts)  $\int_{-1}^2 (3x^2 - 6x + 1) dx$ .

b) (4 pts)  $\int_1^3 (x - 2)^7 dx$ .

c) (4 pts)  $\int_{-2}^2 \frac{x^3}{\sqrt{1+x^2}} dx$ .



9. a) (6 pts) Find an expression for the area between the graphs of  $f(x) = x^2$  and  $g(x) = 3x - 2$  on the interval  $[0, 4]$ . Write the answer as a sum of integrals without absolute values—do *not* evaluate the integrals.

b) (6 pts) Find the average value of the function  $f(x) = \sqrt{x}$  on the interval  $[0, 4]$ .

10. BONUS PROBLEM. (12 pts) Let

$$g(x) = \begin{cases} 0 & \text{if } x = \frac{1}{n} \text{ for some } n \in \\ 1 & \text{if } x \neq \frac{1}{n} \text{ for some } n \in \end{cases}$$

Prove that  $g$  is integrable on  $[0, 1]$  and find the value of  $\int_0^1 g(x) dx$ .