Math 165: Honors Calculus I Name:____________________________
Final Exam *Dec. 19, 1994*

There are 15 problems worth from 5 to 25 points for a total of 170 points.

1. (25 pts) Give complete definitions for the following concepts.

   a) A function f is integrable on $[a, b]$.

   b) $\lim_{x \to p} f(x) = A$.

   c) A function f is continuous at p.

   d) A function f is differentiable at p.

   e) f is a convex function on $[a, b]$.

2. (25 pts) State the following theorems precisely.

   a) PRINCIPLE OF MATHEMATICAL INDUCTION.

   b) BINOMIAL THEOREM.

   c) EXPANSION OR CONTRACTION OF THE INTERVAL OF INTEGRATION.

   d) INTERMEDIATE VALUE THEOREM.

   e) SECOND DERIVATIVE TEST FOR EXTREMA.

3. (20 pts) Calculate the following.

a) $\int_0^2 [x^2]\, dx$ where $[u]$ is the greatest integer $\leq u$.

b) $\int_1^x t^2 + (t-1)^{1/2}\, dt,\ x \geq 1.$

c) The average value of the function $f(x) = x^3$ on the interval $[-1, 2]$.

d) $\int_{-1}^{1} \frac{x}{x^4 + 4}\, dx$

4. (20 pts) Compute the following limits or prove they do not exist.

a) $\lim_{x\to 1}(x^2 - x)\left(1 - \cos\frac{1}{x-1}\right)$.

b) $\lim_{x\to 2^-}\frac{\sqrt{x^3 - 4x^2 + 4x}}{x-2}$.

c) $\lim_{x\to 0}\frac{(1+x^2)^n - (1-x^2)^n}{x^2}$ where n is a fixed positive integer.

d) $\lim_{x\to 0}\frac{\sin(3x)}{\sin(x)}$.

5. (20 pts) Compute the following derivatives.

a) $\frac{d}{dx}\left[x^2(1-x)^{12}\right]$.

b) $\frac{d}{dx}\sqrt{1+x^2}$

c) $\frac{d}{dx}\sin(\sin(x^3))$

d) $\frac{dy}{dx}$ where y is defined implicitly by the equation $y^3+xy+1=0$.

6. (5 pts) Find the largest interval containing $x = 1/2$ on which the function $f(x) = (x^2 - 1)^2$ has an inverse. Give a formula for the corresponding inverse function, $f^{-1}(x)$, as a function of x.

7. (5 pts) Find an expression for the area between the graphs of the functions $f(x) = x^2 - 2x$ and $g(x) = 1 - x$ on the interval $[0, 3]$. Write the answer as a sum of integrals without absolute values—do not evaluate the integrals.

8. (5 pts) Find the equation of the line tangent to the curve defined by $y = x^4 - x^3 + x^2 - x + 1$ at the point $(1, 1)$.

9. (5 pts) Consider the Fibonacci sequence, 1, 1, 2, 3, 5, 8, …, where the next term is the sum of the previous two. Let a_n be the nth term of this sequence so that $a_1 = 1$, $a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n$, $n \geq 1$. Use induction to prove that

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

You may find the following useful:

$$\left(\frac{2}{1\pm\sqrt{5}}\right)^2 + \left(\frac{2}{1\pm\sqrt{5}}\right) = 1$$

10. (5 pts) Prove that the equation $x\cos(x) = 1 - x^2$ has a solution in the interval $[0, \pi]$.

11. (5 pts) Use the *definition of a limit* to prove that $\lim_{x \to 1} x^2 = 1$.

12. (5 pts) Let

$$f(x) = \begin{cases} 6\sqrt{x} - 5, & \text{for } x < 1 \\ x^3, & \text{for } x \geq 1 \end{cases}$$

Prove that $f'(1) = 3$. You must use the definition of a derivative for this and should examine one-sided limits.

13. (5 pts) If a and b are legs of a right triangle whose hypotenuse is 1, find the largest value of $2a + b$.

14. (10 pts) Let $f(x) = 3x^4 + 4x^3 - 12x^2$.

a) Determine the intervals on which f is increasing and decreasing.

b) Determine the relative extrema of f.

c) Determine the intervals on which f is convex and concave.

d) Sketch the graph of f.

15. (10 pts) State and prove the MEAN VALUE THEOREM FOR DERIVATIVES. You may use ROLLE'S THEOREM.